

Introducción al método de diferencias finitas

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Capítulo 1

Diferenciación numérica

Capítulo 2

Ecuaciones diferenciales parciales

Capítulo 3

Ecuación elíptica

Capítulo 4

Ecuación hiperbólica

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0$$

4.1. Métodos explícitos

4.1.1. FTCS

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} + u \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x} = 0$$

Reordenando

$$T(t + \Delta t) - T(t) + \frac{u\Delta t}{2\Delta x} [T(x + \Delta x) - T(x - \Delta x)] = 0$$

Se reemplaza por C , el número de Courant-Friedrich-Levy

$$T(t + \Delta t) - T(t) + \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t) - \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t, x) - \frac{C}{2} [T(t, x + \Delta x) - T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) - \frac{C}{2} [T(n, j + 1) - T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = \frac{C}{2} T(n, j - 1) + T(n, j) - \frac{C}{2} T(n, j + 1) \tag{4.1}$$

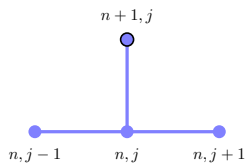


Figura 4.1: Esquema FTCS

Ejemplo 4.1.1.

Resolver la siguiente ecuación diferencial

$$\begin{aligned} \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} &= 0 \\ T(x, 0) &= 0 \\ T(0, t) &= 10 \\ T(6, t) &= 3 \end{aligned}$$

Usando el esquema FTCS con $\Delta x = 1$ y $\Delta t = 1$

Solución

Verificando el número Courant

$$C = \frac{u\Delta t}{\Delta x} = \frac{1 \cdot 1}{1} = 1 \leq 1$$

Dibujando la malla

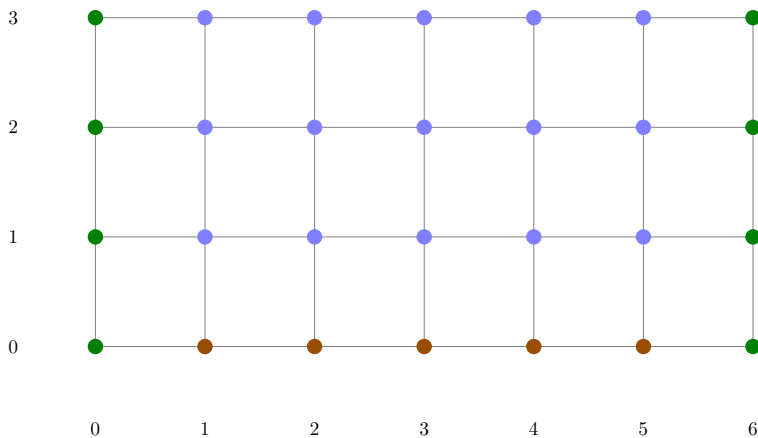


Figura 4.2: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

Figura 4.3: Matriz solución sin valores

Rellenando los valores conocidos

3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	3
1	10	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 4.4: Matriz solución para $t = 0$

Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

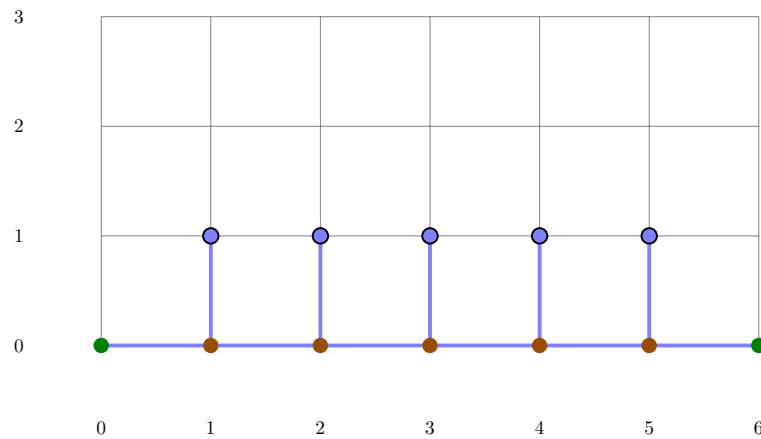


Figura 4.5: Movimiento del esquema para $t = 1$

$$\begin{aligned}
 T(1,1) &= 0.5T(0,0) + T(0,1) - 0.5T(0,2) = 0.5(10) + 0 - 0.5(0) = 5 \\
 T(1,2) &= 0.5T(0,1) + T(0,2) - 0.5T(0,3) = 0.5(0) + 0 - 0.5(0) = 0 \\
 T(1,3) &= 0.5T(0,2) + T(0,3) - 0.5T(0,4) = 0.5(0) + 0 - 0.5(0) = 0 \\
 T(1,4) &= 0.5T(0,3) + T(0,4) - 0.5T(0,5) = 0.5(0) + 0 - 0.5(0) = 0 \\
 T(1,5) &= 0.5T(0,4) + T(0,5) - 0.5T(0,6) = 0.5(0) + 0 - 0.5(3) = -1.5
 \end{aligned}$$

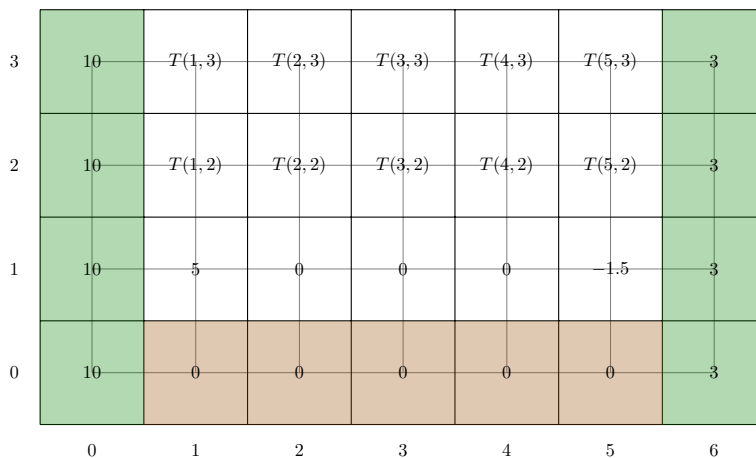


Figura 4.6: Matriz solución para $t = 1$

Para $t = 2$

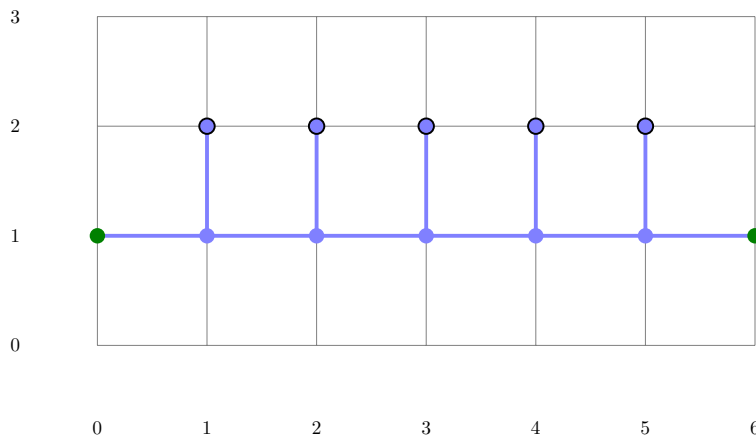


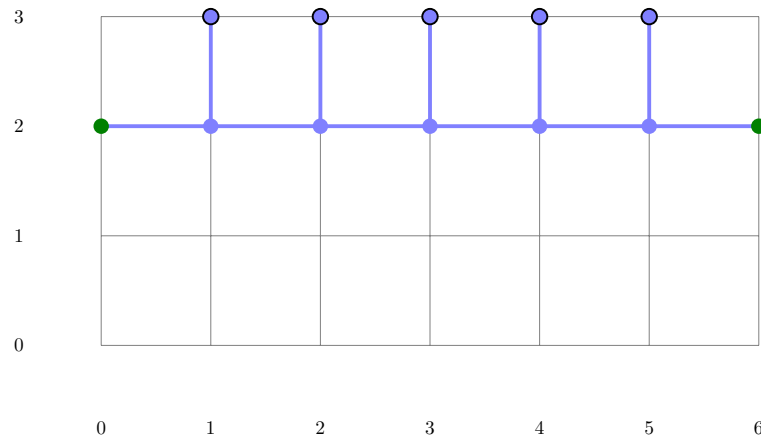
Figura 4.7: Movimiento del esquema para $t = 2$

$$\begin{aligned}
 T(1,1) &= 0.5T(0,0) + T(0,1) - 0.5T(0,2) = 0.5(10) + 5 - 0.5(0) = 10 \\
 T(1,2) &= 0.5T(0,1) + T(0,2) - 0.5T(0,3) = 0.5(5) + 0 - 0.5(0) = 2.5 \\
 T(1,3) &= 0.5T(0,2) + T(0,3) - 0.5T(0,4) = 0.5(0) + 0 - 0.5(0) = 0 \\
 T(1,4) &= 0.5T(0,3) + T(0,4) - 0.5T(0,5) = 0.5(0) + 0 - 0.5(-1.5) = 0.75 \\
 T(1,5) &= 0.5T(0,4) + T(0,5) - 0.5T(0,6) = 0.5(0) - 1.5 - 0.5(3) = -3
 \end{aligned}$$

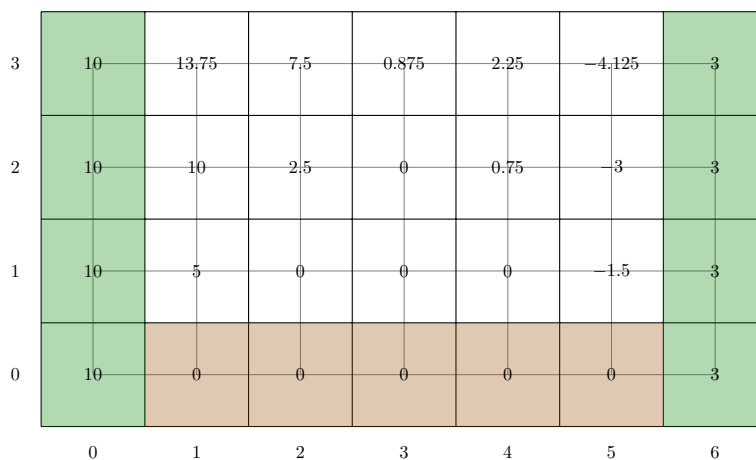
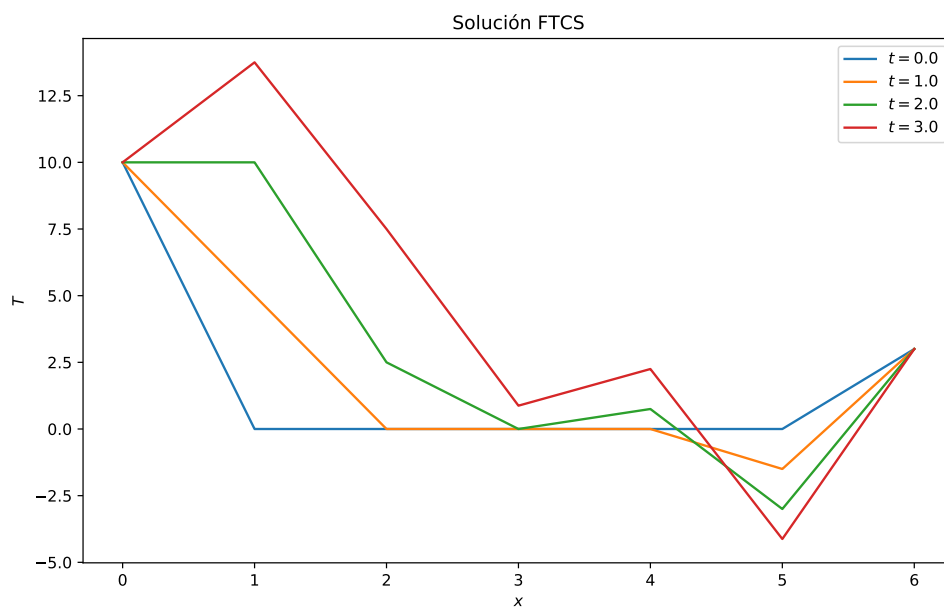
3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	10	2.5	0	0.75	-3	3
1	10	5	0	0	0	-1.5	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 4.8: Matriz solución para $t = 2$

Para $t = 3$

Figura 4.9: Movimiento del esquema para $t = 3$

$$\begin{aligned}
 T(1,1) &= 0.5T(0,0) + T(0,1) - 0.5T(0,2) = 0.5(10) + 10 - 0.5(2.5) = 13.75 \\
 T(1,2) &= 0.5T(0,1) + T(0,2) - 0.5T(0,3) = 0.5(10) + 2.5 - 0.5(0) = 7.5 \\
 T(1,3) &= 0.5T(0,2) + T(0,3) - 0.5T(0,4) = 0.5(2.5) + 0 - 0.5(0.75) = 0.875 \\
 T(1,4) &= 0.5T(0,3) + T(0,4) - 0.5T(0,5) = 0.5(0) + 0.75 - 0.5(-3) = 2.25 \\
 T(1,5) &= 0.5T(0,4) + T(0,5) - 0.5T(0,6) = 0.5(0.75) - 3 - 0.5(3) = -4.125
 \end{aligned}$$

Figura 4.10: Matriz solución para $t = 3$ Figura 4.11: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

4.1.2. FTFS o Downwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación hacia adelante de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x + \Delta x) - T(x)}{\Delta x}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} + u \frac{T(x + \Delta x) - T(x)}{\Delta x} = 0$$

Reordenando

$$T(t + \Delta t) - T(t) + \frac{u\Delta t}{\Delta x} [T(x + \Delta x) - T(x)] = 0$$

Se reemplaza por el término C

$$T(t + \Delta t) - T(t) + C[T(x + \Delta x) - T(x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t) - C[T(x + \Delta x) - T(x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t, x) - C[T(t, x + \Delta x) - T(t, x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) - C[T(n, j + 1) - T(n, j)]$$

Reordenando

$$T(n + 1, j) = (1 + C)T(n, j) - CT(n, j + 1) \quad (4.2)$$

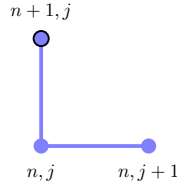


Figura 4.12: Esquema FTFS

4.1.3. FTBS o Upwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación hacia atrás de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x) - T(x - \Delta x)}{\Delta x}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} + u \frac{T(x) - T(x - \Delta x)}{\Delta x} = 0$$

Reordenando

$$T(t + \Delta t) - T(t) + \frac{u\Delta t}{\Delta x}[T(x) - T(x - \Delta x)] = 0$$

Se reemplaza por el término C

$$T(t + \Delta t) - T(t) + C[T(x) - T(x - \Delta x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t) - C[T(x) - T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t, x) - C[T(t, x) - T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) - C[T(n, j) - T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = CT(n, j - 1) + (1 - C)T(n, j) \quad (4.3)$$

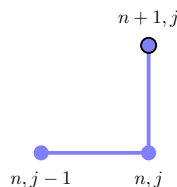


Figura 4.13: Esquema FTBS

4.1.4. CTCS o Leap-Frog

Aproximación centrada de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t - \Delta t)}{2\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t - \Delta t)}{2\Delta t} + u \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x} = 0$$

Reordenando

$$T(t + \Delta t) - T(t - \Delta t) + \frac{u\Delta t}{\Delta x} [T(x + \Delta x) - T(x - \Delta x)] = 0$$

Se reemplaza por el término C

$$T(t + \Delta t) - T(t - \Delta t) + C[T(x + \Delta x) - T(x - \Delta x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t - \Delta t) - C[T(x + \Delta x) - T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t - \Delta t, x) - C[T(t, x + \Delta x) - T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n - 1, j) - C[T(n, j + 1) - T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = T(n - 1, j) - CT(n, j + 1) + CT(n, j - 1) \quad (4.4)$$

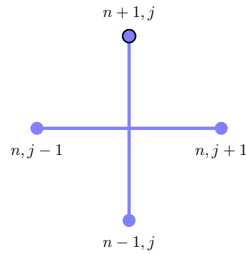


Figura 4.14: Esquema CTCS

4.1.5. Lax

Se deriva de FTCS

$$T(n + 1, j) = T(n, j) - \frac{C}{2} [T(n, j + 1) - T(n, j - 1)]$$

El nodo actual se reemplaza por el promedio de los nodos adyacentes

$$T(n, j) = \frac{T(n, j - 1) + T(n, j + 1)}{2}$$

Reemplazando

$$T(n + 1, j) = \frac{1}{2} [T(n, j - 1) + T(n, j + 1)] - \frac{C}{2} [T(n, j + 1) - T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = \left(\frac{1 + C}{2} \right) T(n, j - 1) + \left(\frac{1 - C}{2} \right) T(n, j + 1) \quad (4.5)$$

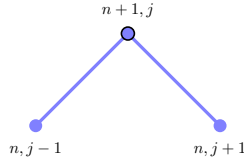


Figura 4.15: Esquema Lax

4.1.6. Lax-Friedrichs

Se deriva de FTCS

$$T(n+1, j) = T(n, j) - \frac{C}{2}[T(n, j+1) - T(n, j-1)]$$

El nodo actual se reemplaza por

$$T(n, j) = T(n, j) + \alpha T(n, j) - \alpha T(n, j)$$

Luego se reemplaza por el promedio ponderado de los nodos adyacentes

$$\alpha T(n, j) = \frac{\alpha T(n, j+1) + \alpha T(n, j-1)}{2}$$

Reemplazando y reordenando

$$T(n, j) = \frac{\alpha}{2}[T(n, j+1) + T(n, j-1)] + (1-\alpha)T(n, j)$$

Reemplazando en el método FTCS

$$T(n+1, j) = \frac{\alpha}{2}[T(n, j+1) + T(n, j-1)] + (1-\alpha)T(n, j) - \frac{C}{2}[T(n, j+1) - T(n, j-1)]$$

Reordenando

$$T(n+1, j) = \left(\frac{\alpha+C}{2}\right)T(n, j-1) + (1-\alpha)T(n, j) + \left(\frac{\alpha-C}{2}\right)T(n, j+1) \quad (4.6)$$

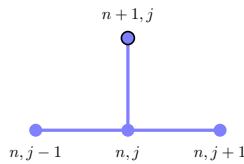


Figura 4.16: Esquema Lax-Friedrichs

4.1.7. Lax-Wendroff

Este método usa la serie de Taylor para mejorar el orden de aproximación

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2$$

Reemplazando $a = t$ y $z - a = \Delta t$

$$T(t + \Delta t, x) = T(t, x) + \Delta t \frac{\partial}{\partial t} T(t, x) + \frac{(\Delta t)^2}{2} \frac{\partial^2}{\partial t^2} T(t, x)$$

Las derivadas son

$$\begin{aligned} \frac{\partial T}{\partial t} &= -u \frac{\partial T}{\partial x} \\ \frac{\partial^2 T}{\partial t^2} &= \frac{\partial}{\partial t} \left(-u \frac{\partial T}{\partial x} \right) = -u \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial x} \right) = -u \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial t} \right) = u^2 \frac{\partial^2 T}{\partial x^2} \end{aligned}$$

Reemplazando

$$T(t + \Delta t, x) = T(t, x) - u\Delta t \frac{\partial}{\partial x} T(t, x) + \frac{(u\Delta t)^2}{2} \frac{\partial^2}{\partial x^2} T(t, x)$$

Aproximaciones centradas de T_x y T_{xx}

$$\begin{aligned} \frac{\partial T}{\partial x} &= \frac{T(t, x + \Delta x) - T(t, x - \Delta x)}{2\Delta x} \\ \frac{\partial^2 T}{\partial x^2} &= \frac{T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)}{(\Delta x)^2} \end{aligned}$$

Reemplazando

$$T(t + \Delta t, x) = T(t, x) - \frac{u\Delta t}{2\Delta x} [T(t, x + \Delta x) - T(t, x - \Delta x)] + \frac{1}{2} \left(\frac{u\Delta t}{\Delta x} \right)^2 [T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Reemplazando por el término C

$$T(t + \Delta t, x) = T(t, x) - \frac{C}{2} [T(t, x + \Delta x) - T(t, x - \Delta x)] + \frac{C^2}{2} [T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) - \frac{C}{2} [T(n, j + 1) - T(n, j - 1)] + \frac{C^2}{2} [T(n, j + 1) - 2T(n, j) + T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = \left(\frac{C^2 + C}{2} \right) T(n, j - 1) - (C^2 - 1) T(n, j) + \left(\frac{C^2 - C}{2} \right) T(n, j + 1)$$

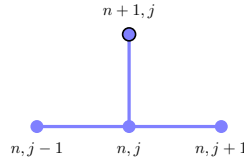


Figura 4.17: Esquema Lax-Wendroff

4.2. Métodos implícitos

4.2.1. FTCS implícito o Laasonen

Se deriva de FTCS

$$T(t + \Delta t) = T(t) - \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)]$$

Reordenando

$$T(t) = T(t + \Delta t) + \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)]$$

Reescribiendo

$$T(t, x) = T(t + \Delta t, x) + \frac{C}{2} [T(t, x + \Delta x) - T(t, x - \Delta x)]$$

Se reemplazará $t = t + \Delta t$ en los elementos multiplicados por $\frac{C}{2}$

$$T(t, x) = T(t + \Delta t, x) + \frac{C}{2} [T(t + \Delta t, x + \Delta x) - T(t + \Delta t, x - \Delta x)]$$

Simplificando

$$T(t, x) = -\frac{C}{2} T(t + \Delta t, x - \Delta x) + T(t + \Delta t, x) + \frac{C}{2} T(t + \Delta t, x + \Delta x)$$

Intercambiando por los índices del mallado

$$T(n, j) = -\frac{C}{2}T(n+1, j-1) + T(n+1, j) + \frac{C}{2}T(n+1, j+1)$$

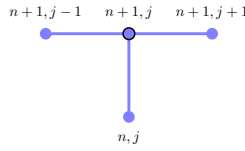


Figura 4.18: Esquema FTCS implícito

Para generalizar el método se usará tres pasos en el espacio

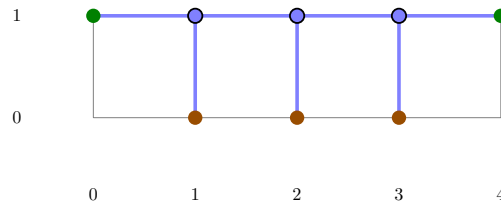


Figura 4.19: Movimiento del esquema para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned} -\frac{C}{2}T[1, 0] + T[1, 1] + \frac{C}{2}T[1, 2] &= T[0, 1] \\ -\frac{C}{2}T[1, 1] + T[1, 2] + \frac{C}{2}T[1, 3] &= T[0, 2] \\ -\frac{C}{2}T[1, 2] + T[1, 3] + \frac{C}{2}T[1, 4] &= T[0, 3] \end{aligned}$$

Reescribiendo

$$\begin{aligned} -\frac{C}{2}T[1, 0] + T[1, 1] + \frac{C}{2}T[1, 2] + 0T[1, 3] + 0T[1, 4] &= T[0, 1] \\ 0T[1, 0] - \frac{C}{2}T[1, 1] + T[1, 2] + \frac{C}{2}T[1, 3] + 0T[1, 4] &= T[0, 2] \\ 0T[1, 0] + 0T[1, 1] - \frac{C}{2}T[1, 2] + T[1, 3] + \frac{C}{2}T[1, 4] &= T[0, 3] \end{aligned}$$

Reordenando

$$\begin{aligned} T[1, 1] + \frac{C}{2}T[1, 2] + 0T[1, 3] &= T[0, 1] + \frac{C}{2}T[1, 0] + 0T[1, 4] \\ -\frac{C}{2}T[1, 1] + T[1, 2] + \frac{C}{2}T[1, 3] &= T[0, 2] + 0T[1, 0] + 0T[1, 4] \\ 0T[1, 1] - \frac{C}{2}T[1, 2] + T[1, 3] &= T[0, 3] + 0T[1, 0] - \frac{C}{2}T[1, 4] \end{aligned}$$

Simplificando

$$\begin{aligned} T[1, 1] + \frac{C}{2}T[1, 2] + 0T[1, 3] &= T[0, 1] + \frac{C}{2}T[1, 0] \\ -\frac{C}{2}T[1, 1] + T[1, 2] + \frac{C}{2}T[1, 3] &= T[0, 2] \\ 0T[1, 1] - \frac{C}{2}T[1, 2] + T[1, 3] &= T[0, 3] - \frac{C}{2}T[1, 4] \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1 & \frac{C}{2} & 0 \\ -\frac{C}{2} & 1 & \frac{C}{2} \\ 0 & -\frac{C}{2} & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \frac{C}{2} \begin{bmatrix} T[1,0] \\ 0 \\ 0 \end{bmatrix} - \frac{C}{2} \begin{bmatrix} 0 \\ 0 \\ T[1,4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1, x]$, también puede escribirse como

$$\begin{bmatrix} 1 & \frac{C}{2} & 0 \\ -\frac{C}{2} & 1 & \frac{C}{2} \\ 0 & -\frac{C}{2} & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} T[0,1] + \frac{C}{2}T[1,0] \\ T[0,2] \\ T[0,3] - \frac{C}{2}T[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix} 1 & \frac{C}{2} & 0 & \dots & \dots & \dots & 0 \\ -\frac{C}{2} & 1 & \frac{C}{2} & \ddots & & & \vdots \\ 0 & -\frac{C}{2} & 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & \frac{C}{2} & 0 \\ \vdots & & & \ddots & -\frac{C}{2} & 1 & \frac{C}{2} \\ 0 & \dots & \dots & \dots & 0 & -\frac{C}{2} & 1 \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} = \begin{bmatrix} T[n,1] + \frac{C}{2}T[n+1,0] \\ T[n,2] \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] - \frac{C}{2}T[n+1,s+1] \end{bmatrix} \quad (4.7)$$

Ejemplo 4.2.1.

Resolver la siguiente ecuación diferencial

$$\begin{aligned} \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} &= 0 \\ T(x, 0) &= 0 \\ T(0, t) &= 10 \\ T(6, t) &= 3 \end{aligned}$$

Usando el esquema FTCS con $\Delta x = 1$ y $\Delta t = 1$

Solución

Verificando el número de Courant

$$C = \frac{u\Delta t}{\Delta x} = \frac{1 \cdot 1}{1} = 1 \leq 1$$

Dibujando la malla

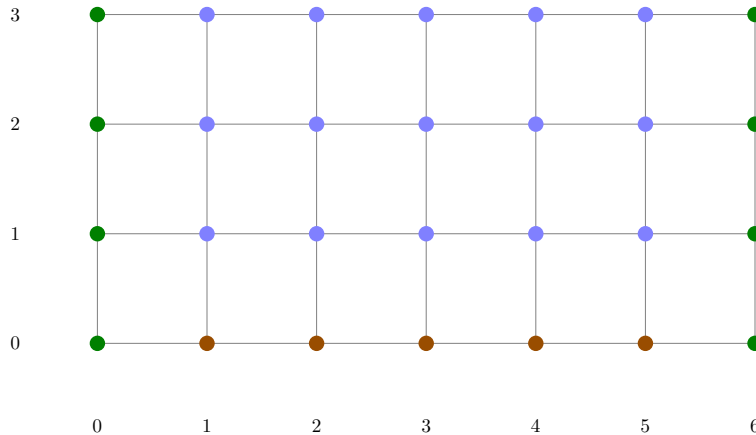


Figura 4.20: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

Figura 4.21: Matriz solución sin valores

Rellenando los valores conocidos

3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	3
1	10	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 4.22: Matriz solución para $t = 0$

Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

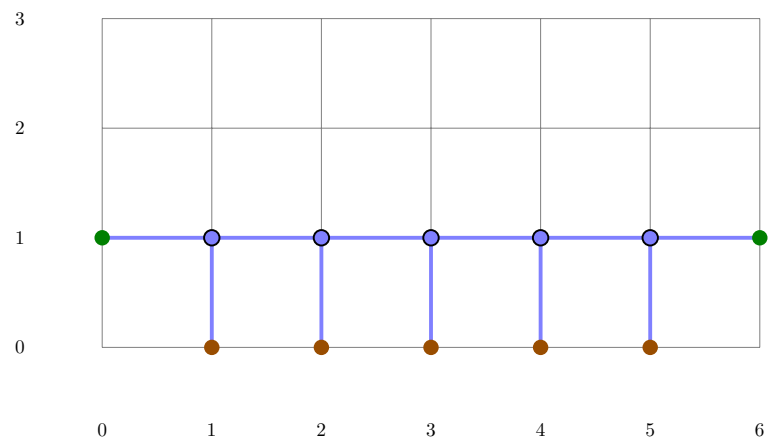


Figura 4.23: Movimiento del esquema para $t = 1$

Usando la fórmula

$$\begin{bmatrix} 1 & \frac{C}{2} & 0 & 0 & 0 \\ -\frac{C}{2} & 1 & \frac{C}{2} & 0 & 0 \\ 0 & -\frac{C}{2} & 1 & \frac{C}{2} & 0 \\ 0 & 0 & -\frac{C}{2} & 1 & \frac{C}{2} \\ 0 & 0 & 0 & -\frac{C}{2} & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} T[0,1] + \frac{C}{2}T[1,0] \\ T[0,2] \\ T[0,3] \\ T[0,4] \\ T[0,5] - \frac{C}{2}T[1,6] \end{bmatrix}$$

Reemplazando valores

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 0 + 0.5(10) \\ 0 \\ 0 \\ 0 \\ 0 - 0.5(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -1.5 \end{bmatrix} \quad T_1 = \begin{bmatrix} 4.1 \\ 1.8 \\ 0.5 \\ 0.8 \\ -1.1 \end{bmatrix}$$

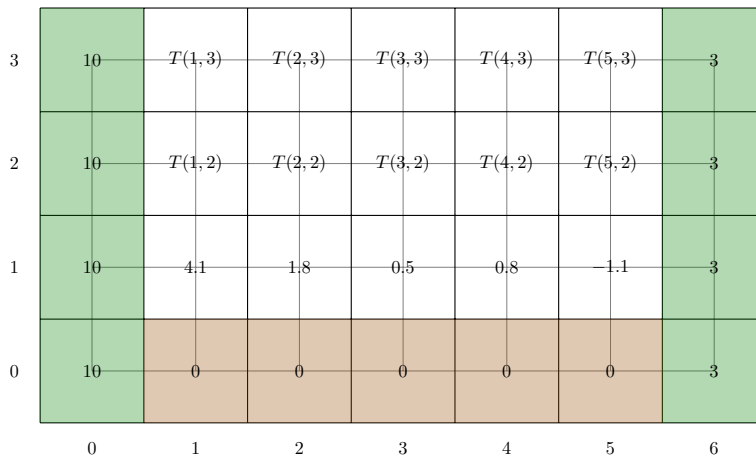


Figura 4.24: Matriz solución para $t = 1$

Para $t = 2$

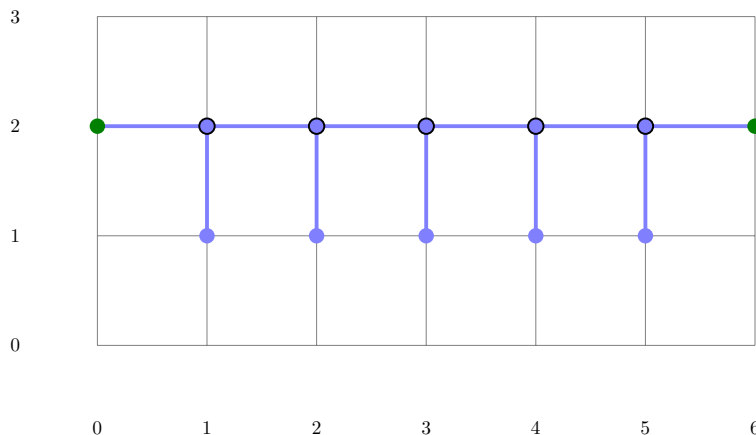


Figura 4.25: Movimiento del esquema para $t = 2$

Reemplazando valores

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 4.1 + 0.5(10) \\ 1.8 \\ 0.5 \\ 0.8 \\ -1.1 - 0.5(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 9.1 \\ 1.8 \\ 0.5 \\ 0.8 \\ -2.6 \end{bmatrix} \quad T_2 = \begin{bmatrix} 6.874 \\ 4.451 \\ 1.571 \\ 2.308 \\ -1.446 \end{bmatrix}$$

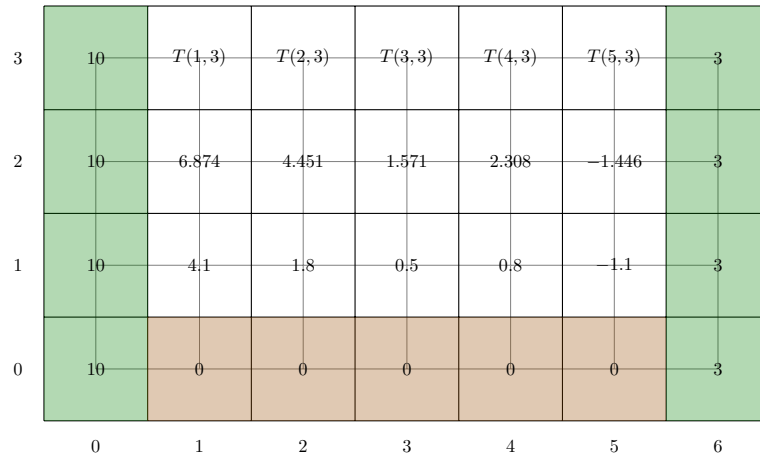


Figura 4.26: Matriz solución para $t = 2$

Para $t = 3$

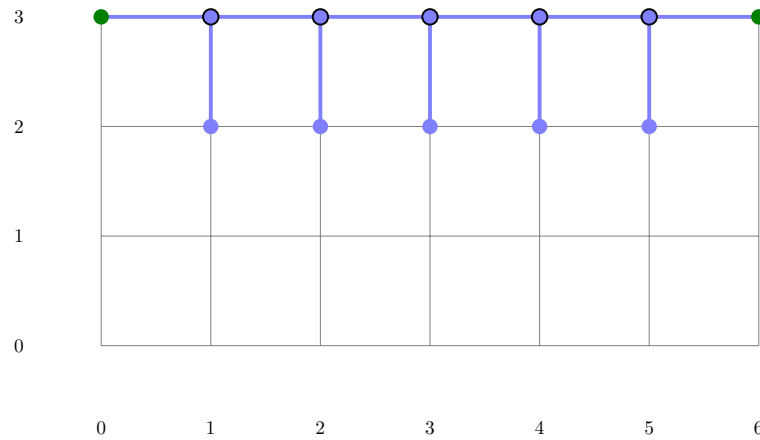


Figura 4.27: Movimiento del esquema para $t = 3$

Reemplazando valores

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 6.874 + 0.5(10) \\ 4.451 \\ 1.571 \\ 2.308 \\ -1.446 - 0.5(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 11.874 \\ 4.451 \\ 1.571 \\ 2.308 \\ -2.946 \end{bmatrix} \quad T_3 = \begin{bmatrix} 8.321 \\ 7.107 \\ 3.010 \\ 4.229 \\ -0.831 \end{bmatrix}$$

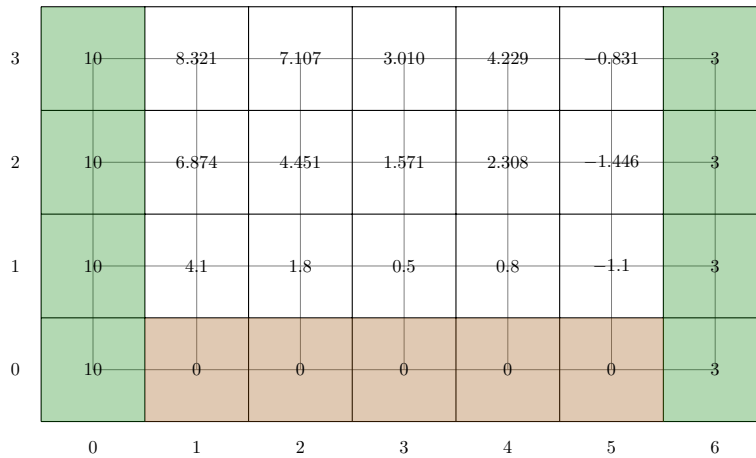


Figura 4.28: Matriz solución para $t = 3$

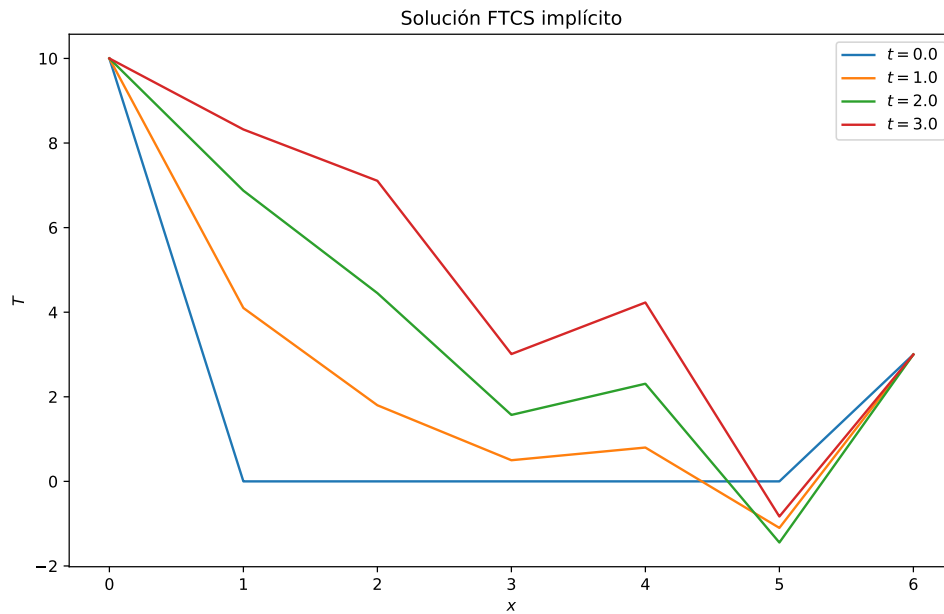


Figura 4.29: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

4.2.2. BTCS

Aproximación hacia atrás de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t) - T(t - \Delta t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x}$$

Reemplazando

$$\frac{T(t) - T(t - \Delta t)}{\Delta t} + u \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x} = 0$$

Reordenando

$$T(t) - T(t - \Delta t) + \frac{u\Delta t}{2\Delta x} [T(x + \Delta x) - T(x - \Delta x)] = 0$$

Se reemplaza por el término C

$$T(t) - T(t - \Delta t) + \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)] = 0$$

Reordenando

$$T(t) = T(t - \Delta t) - \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t, x) = T(t - \Delta t, x) - \frac{C}{2} [T(t, x + \Delta x) - T(t, x - \Delta x)]$$

Intercambiando por los índices del malla

$$T(n, j) = T(n - 1, j) - \frac{C}{2} [T(n, j + 1) - T(n, j - 1)]$$

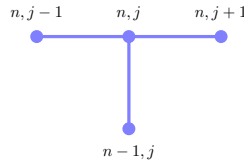


Figura 4.30: Esquema BTCS

4.2.3. Crank-Nicolson

Puede derivarse de métodos FTCS

$$T(t + \Delta t, x) = T(t, x) - \frac{C}{2} [T(t, x + \Delta x) - T(t, x - \Delta x)] \quad \text{explícito}$$

$$T(t, x) = T(t + \Delta t, x) + \frac{C}{2} [T(t + \Delta t, x + \Delta x) - T(t + \Delta t, x - \Delta x)] \quad \text{implícito}$$

Reordenando

$$T(t + \Delta t, x) - T(t, x) = -\frac{C}{2} [T(t, x + \Delta x) - T(t, x - \Delta x)]$$

$$T(t + \Delta t, x) - T(t, x) = -\frac{C}{2} [T(t + \Delta t, x + \Delta x) - T(t + \Delta t, x - \Delta x)]$$

Multiplicando por $\frac{1}{2}$ y sumando

$$T(t + \Delta t, x) - T(t, x) = -\frac{C}{4} [T(t, x + \Delta x) - T(t, x - \Delta x)] - \frac{C}{4} [T(t + \Delta t, x + \Delta x) - T(t + \Delta t, x - \Delta x)]$$

Simplificando

$$T(t + \Delta t, x) - T(t, x) = -\frac{C}{4} [T(t, x + \Delta x) - T(t, x - \Delta x) + T(t + \Delta t, x + \Delta x) - T(t + \Delta t, x - \Delta x)]$$

Reordenando

$$-\frac{C}{4} T(t, x - \Delta x) - T(t, x) + \frac{C}{4} T(t, x + \Delta x) = \frac{C}{4} T(t + \Delta t, x - \Delta x) - T(t + \Delta t, x) - \frac{C}{4} T(t + \Delta t, x + \Delta x)$$

Intercambiando por los índices del mallado

$$-\frac{C}{4}T(n, j-1) - T(n, j) + \frac{C}{4}T(n, j+1) = \frac{C}{4}T(n+1, j-1) - T(n+1, j) - \frac{C}{4}T(n+1, j+1)$$

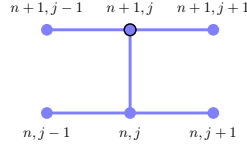


Figura 4.31: Esquema Crank-Nicolson

Para generalizar el método se usará tres pasos en el espacio

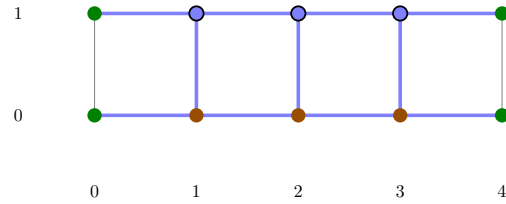


Figura 4.32: Movimiento del esquema para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned} \frac{C}{4}T[1, 0] - T[1, 1] - \frac{C}{4}T[1, 2] &= -\frac{C}{4}T[0, 0] - T[0, 1] + \frac{C}{4}T[0, 2] \\ \frac{C}{4}T[1, 1] - T[1, 2] - \frac{C}{4}T[1, 3] &= -\frac{C}{4}T[0, 1] - T[0, 2] + \frac{C}{4}T[0, 3] \\ \frac{C}{4}T[1, 2] - T[1, 3] - \frac{C}{4}T[1, 4] &= -\frac{C}{4}T[0, 2] - T[0, 3] + \frac{C}{4}T[0, 4] \end{aligned}$$

Reordenando

$$\begin{aligned} -T[1, 1] - \frac{C}{4}T[1, 2] + 0T[1, 3] &= -T[0, 1] + \frac{C}{4}T[0, 2] + 0T[0, 3] - \frac{C}{4}T[0, 0] - \frac{C}{4}T[1, 0] \\ \frac{C}{4}T[1, 1] - T[1, 2] - \frac{C}{4}T[1, 3] &= -\frac{C}{4}T[0, 1] - T[0, 2] + \frac{C}{4}T[0, 3] \\ 0T[1, 1] + \frac{C}{4}T[1, 2] - T[1, 3] &= T[0, 1] - \frac{C}{4}T[0, 2] - T[0, 3] + \frac{C}{4}T[0, 4] + \frac{C}{4}T[1, 4] \end{aligned}$$

En forma matricial

$$\begin{bmatrix} -1 & -\frac{C}{4} & 0 \\ \frac{C}{4} & -1 & -\frac{C}{4} \\ 0 & \frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[1, 1] \\ T[1, 2] \\ T[1, 3] \end{bmatrix} = \begin{bmatrix} -1 & \frac{C}{4} & 0 \\ -\frac{C}{4} & -1 & \frac{C}{4} \\ 0 & -\frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[0, 1] \\ T[0, 2] \\ T[0, 3] \end{bmatrix} - \frac{C}{4} \begin{bmatrix} T[0, 0] \\ 0 \\ 0 \end{bmatrix} + \frac{C}{4} \begin{bmatrix} 0 \\ 0 \\ T[0, 4] \end{bmatrix} - \frac{C}{4} \begin{bmatrix} T[1, 0] \\ 0 \\ 0 \end{bmatrix} + \frac{C}{4} \begin{bmatrix} 0 \\ 0 \\ T[1, 4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1, x]$, también puede escribirse como

$$\begin{bmatrix} -1 & -\frac{C}{4} & 0 \\ \frac{C}{4} & -1 & -\frac{C}{4} \\ 0 & \frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[1, 1] \\ T[1, 2] \\ T[1, 3] \end{bmatrix} = \begin{bmatrix} -1 & \frac{C}{4} & 0 \\ -\frac{C}{4} & -1 & \frac{C}{4} \\ 0 & -\frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[0, 1] \\ T[0, 2] \\ T[0, 3] \end{bmatrix} + \frac{C}{4} \begin{bmatrix} -T[0, 0] - T[1, 0] \\ 0 \\ T[0, 4] + T[1, 4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix} -1 & -\frac{C}{4} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{C}{4} & -1 & -\frac{C}{4} & \ddots & & & & \vdots \\ 0 & \frac{C}{4} & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & -1 & -\frac{C}{4} & 0 \\ \vdots & & & & \ddots & \frac{C}{4} & -1 & -\frac{C}{4} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} \quad (4.8)$$

$$= \begin{bmatrix} -1 & \frac{C}{4} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\frac{C}{4} & -1 & \frac{C}{4} & \ddots & & & & \vdots \\ 0 & -\frac{C}{4} & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & -1 & \frac{C}{4} & 0 \\ \vdots & & & & \ddots & -\frac{C}{4} & -1 & \frac{C}{4} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] \end{bmatrix} + \frac{C}{4} \begin{bmatrix} -T[n,0] - T[n+1,0] \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ T[n,s+1] + T[n+1,s+1] \end{bmatrix}$$

4.2.4. θ

Reemplazando en el método de Crank-Nicolson, $\theta = \frac{1}{4}$

$$-\theta CT(n, j-1) - T(n, j) + \theta CT(n, j+1) = \theta CT(n+1, j-1) - T(n+1, j) - \theta CT(n+1, j+1)$$

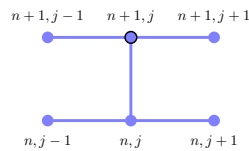


Figura 4.33: Esquema θ

Realizando los mismos pasos para los métodos anteriores, se obtiene

$$\begin{bmatrix} -1 & -\theta C & 0 \\ \theta C & -1 & -\theta C \\ 0 & \theta C & -1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} -1 & \theta C & 0 \\ -\theta C & -1 & \theta C \\ 0 & -\theta C & -1 \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \theta C \begin{bmatrix} -T[0,0] - T[1,0] \\ 0 \\ T[0,4] + T[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{aligned}
 & \begin{bmatrix} -1 & -\theta C & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \theta C & -1 & -\theta C & \ddots & & & & \vdots \\ 0 & \theta C & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & -1 & -\theta C & 0 \\ \vdots & & & & \ddots & \theta C & -1 & -\theta C \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \theta C & -1 \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} \\
 & = \begin{bmatrix} -1 & \theta C & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\theta C & -1 & \theta C & \ddots & & & & \vdots \\ 0 & -\theta C & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & -1 & \theta C & 0 \\ \vdots & & & & \ddots & -\theta C & -1 & \theta C \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\theta C & -1 \end{bmatrix} \begin{bmatrix} T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] \end{bmatrix} + \theta C \begin{bmatrix} -T[n,0] - T[n+1,0] \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ T[n,s+1] + T[n+1,s+1] \end{bmatrix} \tag{4.9}
 \end{aligned}$$

Capítulo 5

Ecuación parabólica

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

5.1. Métodos explícitos

5.1.1. FTCS

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} - D \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t + \Delta t) - T(t) - \frac{D\Delta t}{\Delta x^2} [T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Se reemplaza por S , el número de difusión

$$T(t + \Delta t) - T(t) - S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t) + S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t, x) + S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) + S[T(n, j + 1) - 2T(n, j) + T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = ST(n, j - 1) + (1 - 2S)T(n, j) + ST(n, j + 1) \tag{5.1}$$

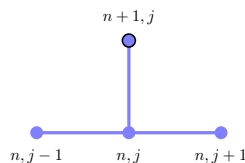


Figura 5.1: Esquema FTCS

Ejemplo 5.1.1.

Resolver la siguiente ecuación diferencial

$$\frac{\partial T}{\partial t} = 0.4 \frac{\partial^2 T}{\partial x^2}$$

$$T(x, 0) = 0$$

$$T(0, t) = 10$$

$$T(6, t) = 3$$

Usando el esquema FTCS con $\Delta x = 1$ y $\Delta t = 1$

Solución

Verificando el número de difusión

$$S = \frac{D\Delta t}{\Delta x^2} = \frac{0.4 \cdot 1}{1^2} = 0.4 \leq 0.5$$

Dibujando la malla

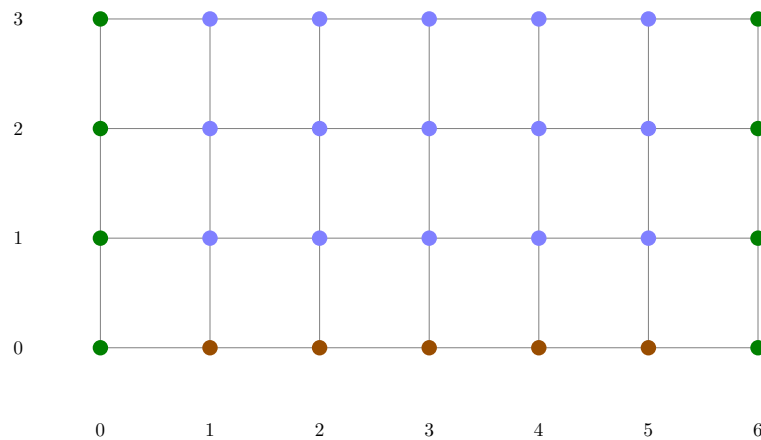


Figura 5.2: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

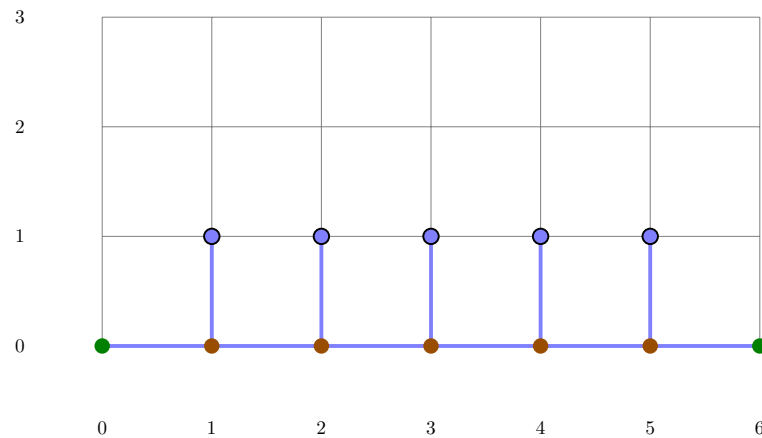
Figura 5.3: Matriz solución sin valores

Rellenando los valores conocidos

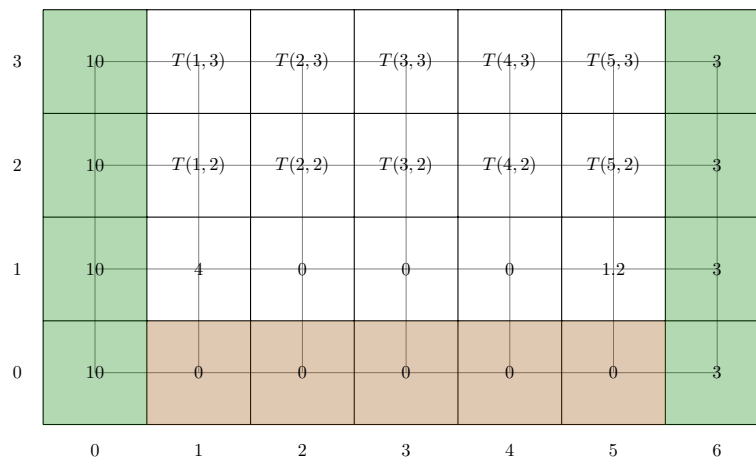
3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	3
1	10	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 5.4: Matriz solución para $t = 0$

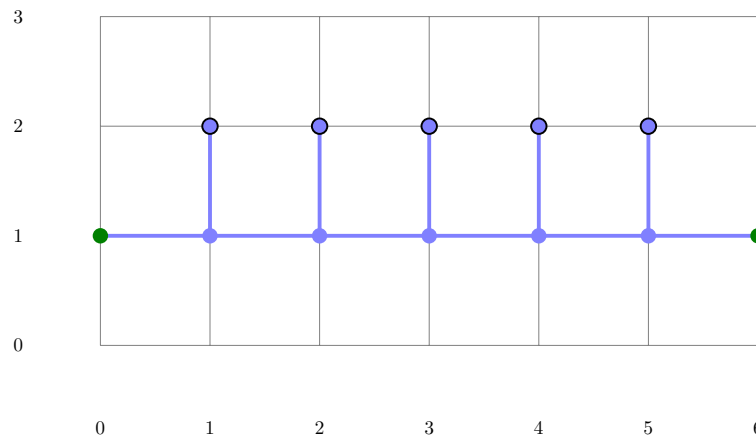
Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

Figura 5.5: Movimiento del esquema para $t = 1$

$$\begin{aligned}
 T(1,1) &= 0.4T(0,0) + 0.2T(0,1) + 0.4T(0,2) = 0.4(10) + 0.2(0) + 0.4(0) = 4 \\
 T(1,2) &= 0.4T(0,1) + 0.2T(0,2) + 0.4T(0,3) = 0.4(0) + 0.2(0) + 0.4(0) = 0 \\
 T(1,3) &= 0.4T(0,2) + 0.2T(0,3) + 0.4T(0,4) = 0.4(0) + 0.2(0) + 0.4(0) = 0 \\
 T(1,4) &= 0.4T(0,3) + 0.2T(0,4) + 0.4T(0,5) = 0.4(0) + 0.2(0) + 0.4(0) = 0 \\
 T(1,5) &= 0.4T(0,4) + 0.2T(0,5) + 0.4T(0,6) = 0.4(0) + 0.2(0) + 0.4(3) = 1.2
 \end{aligned}$$

Figura 5.6: Matriz solución para $t = 1$

Para $t = 2$

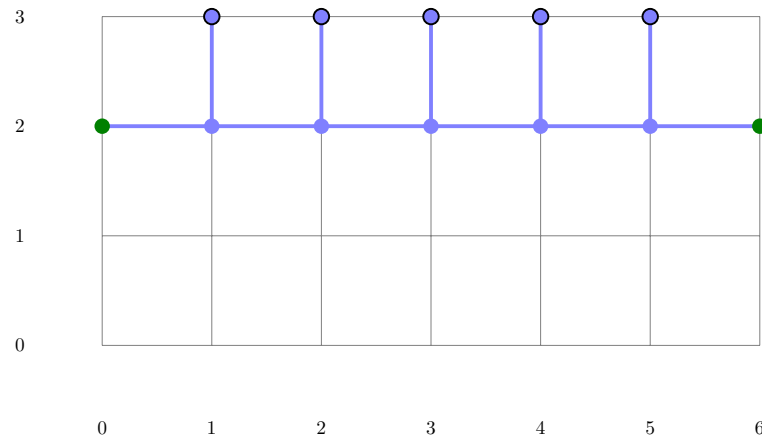
Figura 5.7: Movimiento del esquema para $t = 2$

$$\begin{aligned}
 T(2,1) &= 0.4T(1,0) + 0.2T(1,1) + 0.4T(1,2) = 0.4(10) + 0.2(4) + 0.4(0) = 4.8 \\
 T(2,2) &= 0.4T(1,1) + 0.2T(1,2) + 0.4T(1,3) = 0.4(4) + 0.2(0) + 0.4(0) = 1.6 \\
 T(2,3) &= 0.4T(1,2) + 0.2T(1,3) + 0.4T(1,4) = 0.4(0) + 0.2(0) + 0.4(0) = 0 \\
 T(2,4) &= 0.4T(1,3) + 0.2T(1,4) + 0.4T(1,5) = 0.4(0) + 0.2(0) + 0.4(1.2) = 0.48 \\
 T(2,5) &= 0.4T(1,4) + 0.2T(1,5) + 0.4T(1,6) = 0.4(0) + 0.2(1.2) + 0.4(3) = 1.44
 \end{aligned}$$

3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	4.8	1.6	0	0.48	1.44	3
1	10	4	0	0	0	1.2	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 5.8: Matriz solución para $t = 2$

Para $t = 3$

Figura 5.9: Movimiento del esquema para $t = 3$

$$\begin{aligned}
 T(3,1) &= 0.4T(2,0) + 0.2T(2,1) + 0.4T(2,2) = 0.4(10) + 0.2(4.8) + 0.4(1.6) = 5.6 \\
 T(3,2) &= 0.4T(2,1) + 0.2T(2,2) + 0.4T(2,3) = 0.4(4.8) + 0.2(1.6) + 0.4(0) = 2.24 \\
 T(3,3) &= 0.4T(2,2) + 0.2T(2,3) + 0.4T(2,4) = 0.4(1.6) + 0.2(0) + 0.4(0.48) = 0.832 \\
 T(3,4) &= 0.4T(2,3) + 0.2T(2,4) + 0.4T(2,5) = 0.4(0) + 0.2(0.48) + 0.4(1.44) = 0.672 \\
 T(3,5) &= 0.4T(2,4) + 0.2T(2,5) + 0.4T(2,6) = 0.4(0.48) + 0.2(1.44) + 0.4(3) = 1.68
 \end{aligned}$$

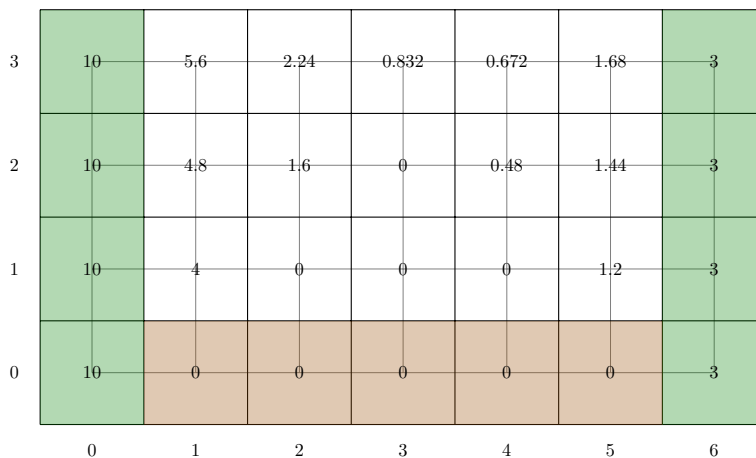


Figura 5.10: Matriz solución para $t = 3$

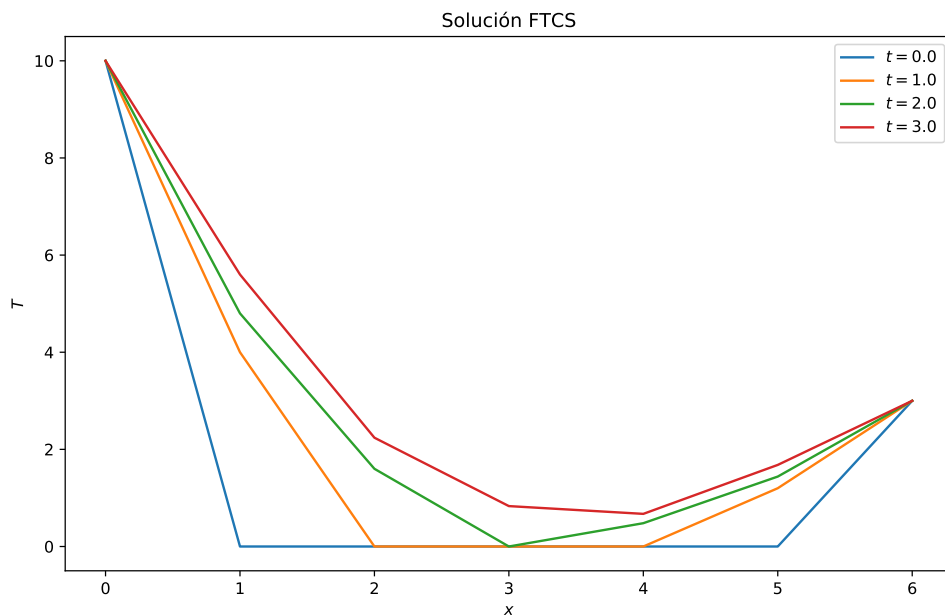


Figura 5.11: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

5.1.2. FTFS o Downwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación hacia adelante de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + 2\Delta x) - 2T(x + \Delta x) + T(x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} - D \frac{T(x + 2\Delta x) - 2T(x + \Delta x) + T(x)}{\Delta x^2} = 0$$

Reordenando

$$T(t + \Delta t) - T(t) - \frac{D\Delta t}{\Delta x^2} [T(x + 2\Delta x) - 2T(x + \Delta x) + T(x)] = 0$$

Se reemplaza por el término S

$$T(t + \Delta t) - T(t) - S[T(x + 2\Delta x) - 2T(x + \Delta x) + T(x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t) + S[T(x + 2\Delta x) - 2T(x + \Delta x) + T(x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t, x) + S[T(t, x + 2\Delta x) - 2T(t, x + \Delta x) + T(t, x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) + S[T(n, j + 2) - 2T(n, j + 1) + T(n, j)]$$

Reordenando

$$T(n + 1, j) = (1 + S)T(n, j) - 2ST(n, j + 1) + ST(n, j + 2) \quad (5.2)$$

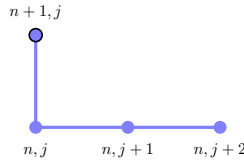


Figura 5.12: Esquema FTFS

5.1.3. FTBS o Upwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación hacia atrás de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x) - 2T(x - \Delta x) + T(x - 2\Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} - D \frac{T(x) - 2T(x - \Delta x) + T(x - 2\Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t + \Delta t) - T(t) - \frac{D\Delta t}{\Delta x^2}[T(x) - 2T(x - \Delta x) + T(x - 2\Delta x)] = 0$$

Se reemplaza por el término S

$$T(t + \Delta t) - T(t) - S[T(x) - 2T(x - \Delta x) + T(x - 2\Delta x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t) + S[T(x) - 2T(x - \Delta x) + T(x - 2\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t, x) + S[T(t, x) - 2T(t, x - \Delta x) + T(t, x - 2\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) + S[T(n, j) - 2T(n, j - 1) + T(n, j - 2)]$$

Reordenando

$$T(n + 1, j) = ST(n, j - 2) - 2ST(n, j - 1) + (1 + S)T(n, j) \quad (5.3)$$

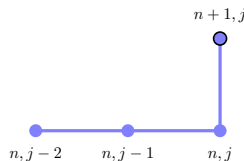


Figura 5.13: Esquema FTBS

5.1.4. CTCS o Leap-Frog

Aproximación centrada de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t - \Delta t)}{2\Delta t}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t - \Delta t)}{2\Delta t} - D \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t + \Delta t) - T(t - \Delta t) - \frac{2D\Delta t}{\Delta x^2} [T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Se reemplaza por el término S

$$T(t + \Delta t) - T(t - \Delta t) - 2S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Reordenando

$$T(t + \Delta t) = T(t - \Delta t) + 2S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t - \Delta t, x) + 2S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n - 1, j) + 2S[T(n, j + 1) - 2T(n, j) + T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = T(n - 1, j) + 2ST(n, j - 1) - 4ST(n, j) + 2ST(n, j + 1) \quad (5.4)$$

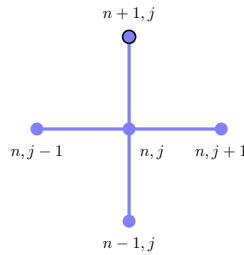


Figura 5.14: Esquema CTCS

5.1.5. Du Fort-Frankel

Se deriva de Leap-Frog

$$T(n + 1, j) = T(n - 1, j) + 2S[T(n, j - 1) - 2T(n, j) + T(n, j + 1)]$$

El nodo actual se reemplaza por el promedio de los nodos adyacentes

$$T(n, j) = \frac{T(n - 1, j) + T(n + 1, j)}{2}$$

Reemplazando

$$T(n + 1, j) = T(n - 1, j) + 2S[T(n, j - 1) - T(n - 1, j) - T(n + 1, j) + T(n, j + 1)]$$

Reordenando

$$T(n+1, j) = \left(\frac{1-2S}{1+2S} \right) T(n-1, j) + \left(\frac{2S}{1+2S} \right) T(n, j-1) + \left(\frac{2S}{1+2S} \right) T(n, j+1) \quad (5.5)$$

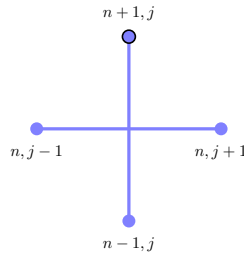


Figura 5.15: Esquema Du Fort-Frankel

5.1.6. Lax

Se deriva de FTCS

$$T(n+1, j) = T(n, j) + S[T(n, j+1) - 2T(n, j) + T(n, j-1)]$$

El nodo actual se reemplaza por el promedio de los nodos adyacentes

$$T(n, j) = \frac{T(n, j-1) + T(n, j) + T(n, j+1)}{3}$$

Reemplazando

$$T(n+1, j) = \frac{1}{3}[T(n, j-1) + T(n, j) + T(n, j+1)] + S[T(n, j+1) - 2T(n, j) + T(n, j-1)]$$

Reordenando

$$T(n+1, j) = \left(\frac{1}{3} + S \right) T(n, j-1) + \left(\frac{1}{3} - 2S \right) T(n, j) + \left(\frac{1}{3} + S \right) T(n, j+1) \quad (5.6)$$

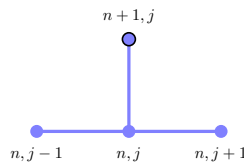


Figura 5.16: Esquema Lax

5.1.7. Lax-Friedrichs

Se deriva de FTCS

$$T(n+1, j) = T(n, j) + S[T(n, j+1) - 2T(n, j) + T(n, j-1)]$$

El nodo actual se reemplaza por

$$T(n, j) = T(n, j) + \alpha T(n, j) - \alpha T(n, j)$$

Luego se reemplaza por el promedio ponderado de los nodos adyacentes

$$\alpha T(n, j) = \frac{\alpha T(n, j+1) + \alpha T(n, j) + \alpha T(n, j-1)}{3}$$

Reemplazando y reordenando

$$T(n, j) = \frac{\alpha}{3}[T(n, j+1) + T(n, j) + T(n, j-1)] + (1-\alpha)T(n, j)$$

Reemplazando

$$T(n+1, j) = \frac{\alpha}{3}[T(n, j+1) + T(n, j) + T(n, j-1)] + (1-\alpha)T(n, j) + S[T(n, j+1) - 2T(n, j) + T(n, j-1)]$$

Reordenando

$$T(n+1, j) = \left(\frac{\alpha}{3} + S\right)T(n, j-1) + \left(1 - \frac{2}{3}\alpha - 2S\right)T(n, j) + \left(\frac{\alpha}{3} + S\right)T(n, j+1) \quad (5.7)$$

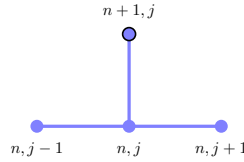


Figura 5.17: Esquema Lax-Friedrichs

5.2. Métodos implícitos

5.2.1. FTCS implícito o Laasonen

Se deriva de FTCS

$$T(t + \Delta t) = T(t) + S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reordenando

$$T(t) = T(t + \Delta t) - S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo

$$T(t, x) = T(t + \Delta t, x) - S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Se reemplazará $t = t + \Delta t$ en los elementos multiplicados por s

$$T(t, x) = T(t + \Delta t, x) - S[T(t + \Delta t, x + \Delta x) - 2T(t + \Delta t, x) + T(t + \Delta t, x - \Delta x)]$$

Simplificando

$$T(t, x) = -ST(t + \Delta t, x + \Delta x) + (1 + 2S)T(t + \Delta t, x) - ST(t + \Delta t, x - \Delta x)$$

Intercambiando por los índices del malla

$$T(n, j) = -ST(n+1, j+1) + (1 + 2S)T(n+1, j) - ST(n+1, j-1)$$

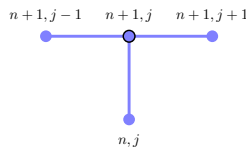
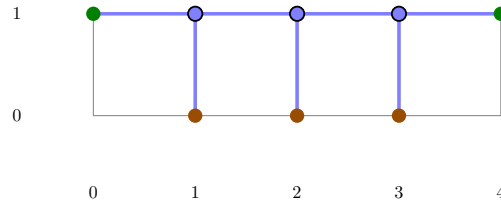


Figura 5.18: Esquema FTCS implícito

Para generalizar el método se usará tres pasos en el espacio

Figura 5.19: Movimiento del esquema para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned} -ST[1, 0] + (1 + 2S)T[1, 1] - ST[1, 2] &= T[0, 1] \\ -ST[1, 1] + (1 + 2S)T[1, 2] - ST[1, 3] &= T[0, 2] \\ -ST[1, 2] + (1 + 2S)T[1, 3] - ST[1, 4] &= T[0, 3] \end{aligned}$$

Reescribiendo

$$\begin{aligned} -ST[1, 0] + (1 + 2S)T[1, 1] - ST[1, 2] + 0T[1, 3] + 0T[1, 4] &= T[0, 1] \\ 0T[1, 0] - ST[1, 1] + (1 + 2S)T[1, 2] - ST[1, 3] + 0T[1, 4] &= T[0, 2] \\ 0T[1, 0] + 0T[1, 1] - ST[1, 2] + (1 + 2S)T[1, 3] - ST[1, 4] &= T[0, 3] \end{aligned}$$

Reordenando

$$\begin{aligned} (1 + 2S)T[1, 1] - ST[1, 2] + 0T[1, 3] &= T[0, 1] + ST[1, 0] + 0T[1, 0] \\ -ST[1, 1] + (1 + 2S)T[1, 2] - ST[1, 3] &= T[0, 2] + 0T[1, 0] + 0T[1, 4] \\ 0T[1, 1] - ST[1, 2] + (1 + 2S)T[1, 3] &= T[0, 3] + 0T[1, 0] + ST[1, 4] \end{aligned}$$

Simplificando

$$\begin{aligned} (1 + 2S)T[1, 1] - ST[1, 2] + 0T[1, 3] &= T[0, 1] + ST[1, 0] \\ -ST[1, 1] + (1 + 2S)T[1, 2] - ST[1, 3] &= T[0, 2] \\ 0T[1, 1] - ST[1, 2] + (1 + 2S)T[1, 3] &= T[0, 3] + ST[1, 4] \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1 + 2S & -S & 0 \\ -S & 1 + 2S & -S \\ 0 & -S & 1 + 2S \end{bmatrix} \begin{bmatrix} T[1, 1] \\ T[1, 2] \\ T[1, 3] \end{bmatrix} = \begin{bmatrix} T[0, 1] \\ T[0, 2] \\ T[0, 3] \end{bmatrix} + S \begin{bmatrix} T[1, 0] \\ 0 \\ T[1, 4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1, x]$, también puede escribirse como

$$\begin{bmatrix} 1 + 2S & -S & 0 \\ -S & 1 + 2S & -S \\ 0 & -S & 1 + 2S \end{bmatrix} \begin{bmatrix} T[1, 1] \\ T[1, 2] \\ T[1, 3] \end{bmatrix} = \begin{bmatrix} T[0, 1] + ST[1, 0] \\ T[0, 2] \\ T[0, 3] + ST[1, 4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix} 1 + 2S & -S & 0 & \cdots & \cdots & \cdots & 0 \\ -S & 1 + 2S & -S & \ddots & & & \vdots \\ 0 & -S & 1 + 2S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 + 2S & -S & 0 \\ \vdots & & & \ddots & -S & 1 + 2S & -S \\ 0 & \cdots & \cdots & \cdots & 0 & -S & 1 + 2S \end{bmatrix} \begin{bmatrix} T[n + 1, 1] \\ T[n + 1, 2] \\ \vdots \\ \vdots \\ T[n + 1, s - 1] \\ T[n + 1, s] \end{bmatrix} = \begin{bmatrix} T[n, 1] + ST[n + 1, 0] \\ T[n, 2] \\ \vdots \\ \vdots \\ T[n, s - 1] \\ T[n, s] + ST[n + 1, s + 1] \end{bmatrix} \quad (5.8)$$

Ejemplo 5.2.1.

Resolver la siguiente ecuación diferencial

$$\frac{\partial T}{\partial t} = 0.4 \frac{\partial^2 T}{\partial x^2}$$

$$T(x, 0) = 0$$

$$T(0, t) = 10$$

$$T(6, t) = 3$$

Usando el esquema FTCS implícito con $\Delta x = 1$ y $\Delta t = 1$

Solución

Verificando el número de difusión

$$S = \frac{D\Delta t}{\Delta x^2} = \frac{0.4 \cdot 1}{1^2} = 0.4 \leq 0.5$$

Dibujando la malla

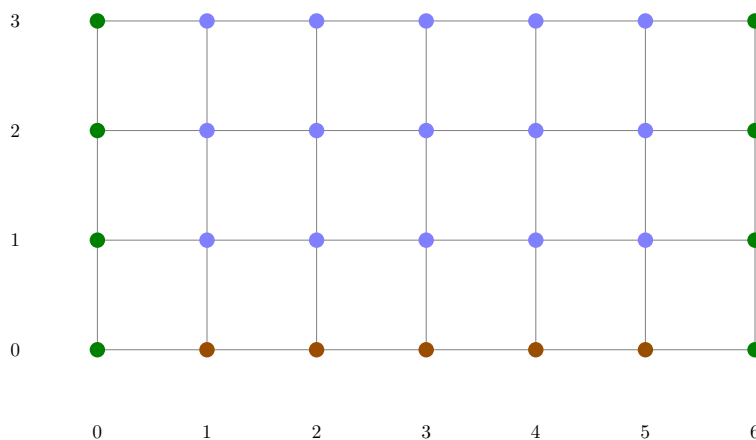


Figura 5.20: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

Figura 5.21: Matriz solución sin valores

Rellenando los valores conocidos

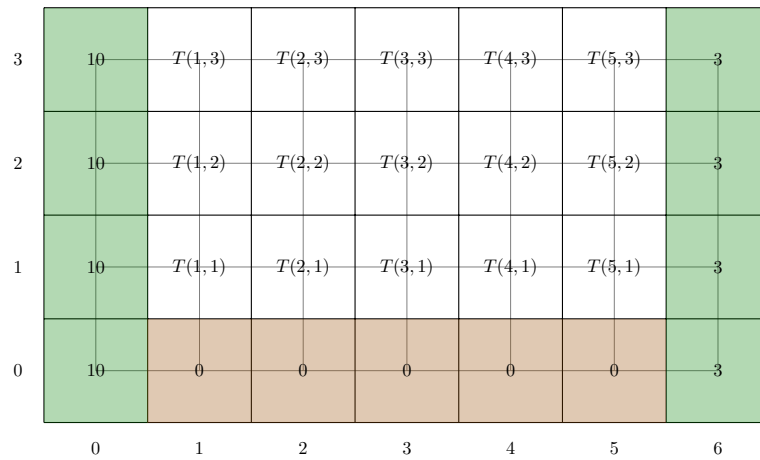


Figura 5.22: Matriz solución para $t = 0$

Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

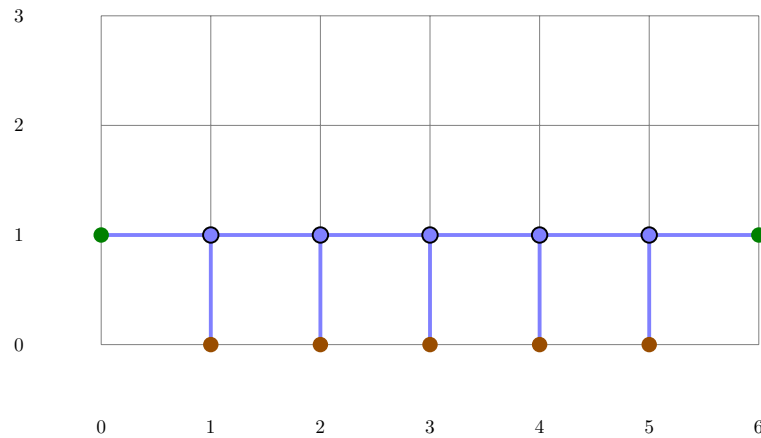


Figura 5.23: Movimiento del esquema para $t = 1$

Usando la fórmula

$$\begin{bmatrix} 1+2S & -S & 0 & 0 & 0 \\ -S & 1+2S & -S & 0 & 0 \\ 0 & -S & 1+2S & -S & 0 \\ 0 & 0 & -S & 1+2S & -S \\ 0 & 0 & 0 & -S & 1+2S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} T[0,1] + ST[1,0] \\ T[0,2] \\ T[0,3] \\ T[0,4] \\ T[0,5] + ST[1,6] \end{bmatrix}$$

Reemplazando valores

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 0 + 0.4(10) \\ 0 \\ 0 \\ 0 \\ 0 + 0.4(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1.2 \end{bmatrix} \quad T_1 = \begin{bmatrix} 2.346 \\ 0.559 \\ 0.167 \\ 0.195 \\ 0.710 \end{bmatrix}$$

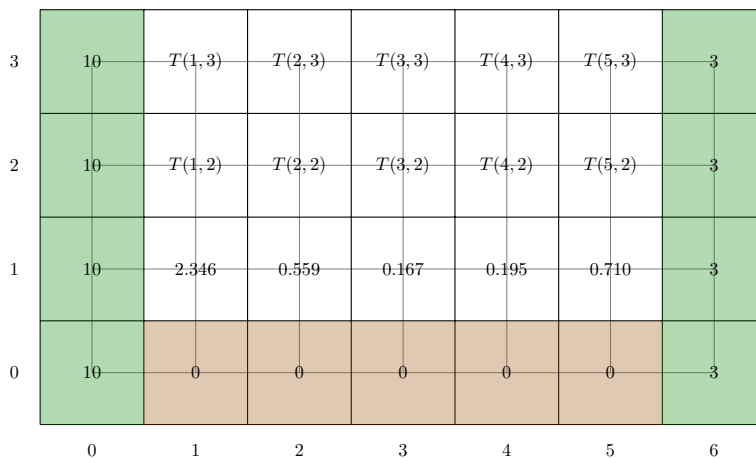


Figura 5.24: Matriz solución para $t = 1$

Para $t = 2$

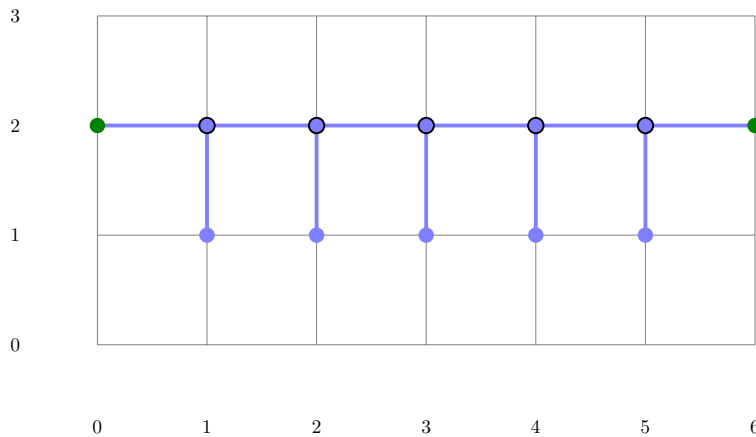


Figura 5.25: Movimiento del esquema para $t = 2$

Reemplazando valores

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 2.346 + 0.4(10) \\ 0.559 \\ 0.167 \\ 0.195 \\ 0.710 + 0.4(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 6.346 \\ 0.559 \\ 0.167 \\ 0.195 \\ 1.910 \end{bmatrix} \quad T_2 = \begin{bmatrix} 3.806 \\ 1.263 \\ 0.479 \\ 0.474 \\ 1.166 \end{bmatrix}$$

3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	3.806	1.263	0.479	0.474	1.166	3
1	10	2.346	0.559	0.167	0.195	0.710	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 5.26: Matriz solución para $t = 2$

Para $t = 3$

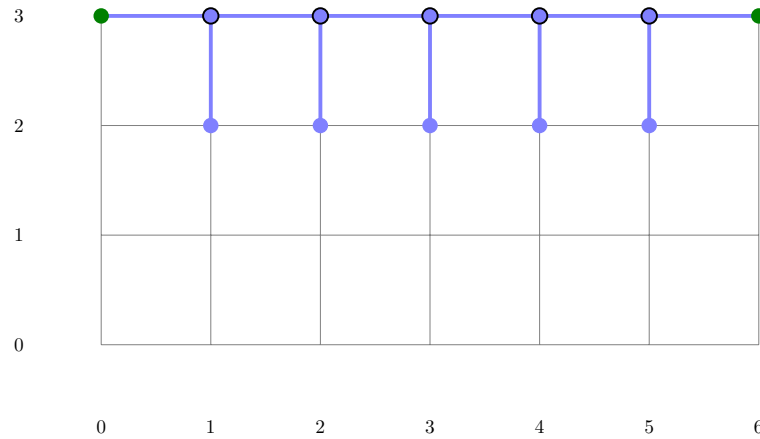


Figura 5.27: Movimiento del esquema para $t = 3$

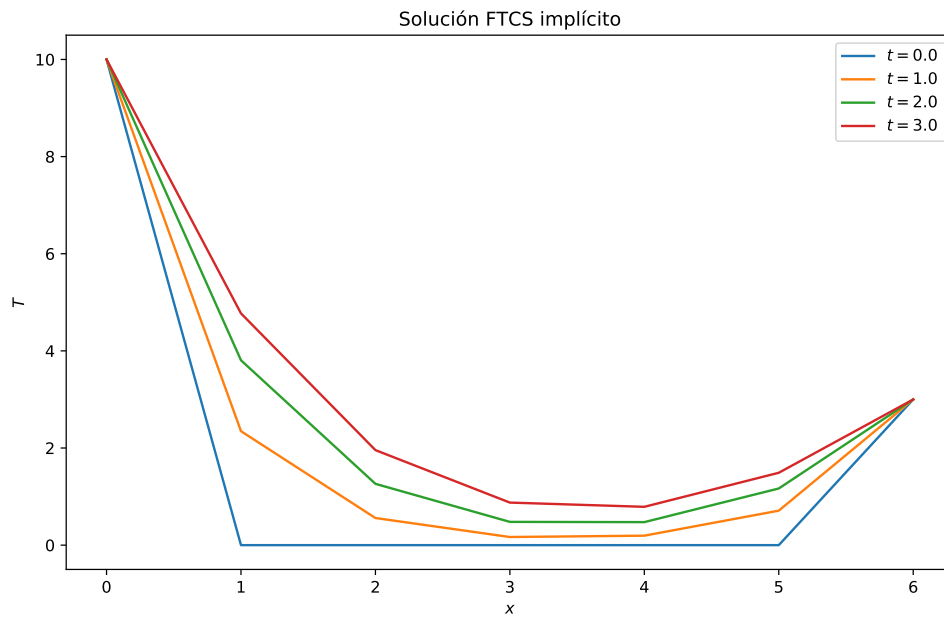
Reemplazando valores

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 3.806 + 0.4(10) \\ 1.263 \\ 0.479 \\ 0.474 \\ 1.166 + 0.4(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 7.806 \\ 1.263 \\ 0.479 \\ 0.474 \\ 2.366 \end{bmatrix} \quad T_3 = \begin{bmatrix} 4.772 \\ 1.956 \\ 0.876 \\ 0.789 \\ 1.490 \end{bmatrix}$$

3	10	4.772	1.956	0.876	0.789	1.490	3
2	10	3.806	1.263	0.479	0.474	1.166	3
1	10	2.346	0.559	0.167	0.195	0.710	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 5.28: Matriz solución para $t = 3$ Figura 5.29: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

5.2.2. BTCS

Aproximación hacia atrás de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t) - T(t - \Delta t)}{\Delta t}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t) - T(t - \Delta t)}{\Delta t} - D \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t) - T(t - \Delta t) - \frac{D\Delta t}{\Delta x^2} [T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Se reemplaza por el término S

$$T(t) - T(t - \Delta t) - S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Reordenando

$$T(t) = T(t - \Delta t) + S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t, x) = T(t - \Delta t, x) + S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n, j) = T(n - 1, j) + S[T(n, j + 1) - 2T(n, j) + T(n, j - 1)]$$

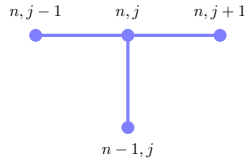


Figura 5.30: Esquema BTCS

5.2.3. Crank-Nicolson

Puede derivarse de métodos FTCS

$$\begin{aligned} T(t + \Delta t, x) &= T(t, x) + S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)] && \text{explícito} \\ T(t, x) &= T(t + \Delta t, x) - S[T(t + \Delta t, x + \Delta x) - 2T(t + \Delta t, x) + T(t + \Delta t, x - \Delta x)] && \text{implícito} \end{aligned}$$

Reordenando

$$\begin{aligned} T(t + \Delta t, x) - T(t, x) &= S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)] \\ T(t + \Delta t, x) - T(t, x) &= S[T(t + \Delta t, x + \Delta x) - 2T(t + \Delta t, x) + T(t + \Delta t, x - \Delta x)] \end{aligned}$$

Multiplicando por $\frac{1}{2}$ y sumando

$$T(t + \Delta t, x) - T(t, x) = \frac{S}{2}[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)] + \frac{S}{2}[T(t + \Delta t, x + \Delta x) - 2T(t + \Delta t, x) + T(t + \Delta t, x - \Delta x)]$$

Simplificando

$$T(t + \Delta t, x) - T(t, x) = \frac{S}{2}[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x) + T(t + \Delta t, x + \Delta x) - 2T(t + \Delta t, x) + T(t + \Delta t, x - \Delta x)]$$

Reordenando

$$\frac{S}{2}T(t, x + \Delta x) + (1 - S)T(t, x) + \frac{S}{2}T(t, x - \Delta x) = -\frac{S}{2}T(t + \Delta t, x + \Delta x) + (1 + S)T(t + \Delta t, x) - \frac{S}{2}T(t + \Delta t, x - \Delta x)$$

Intercambiando por los índices del mallado

$$\frac{S}{2}T(n, j + 1) + (1 - S)T(n, j) + \frac{S}{2}T(n, j - 1) = -\frac{S}{2}T(n + 1, j + 1) + (1 + S)T(n + 1, j) - \frac{S}{2}T(n + 1, j - 1)$$

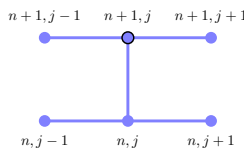
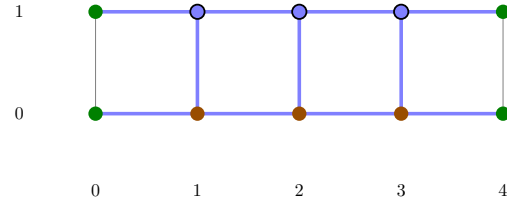


Figura 5.31: Esquema Crank-Nicolson

Para generalizar el método se usará tres pasos en el espacio

Figura 5.32: Movimiento del Esquema para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned} -\frac{S}{2}T[1,0] + (1+S)T[1,1] - \frac{S}{2}T[1,2] &= \frac{S}{2}T[0,0] + (1-S)T[0,1] + \frac{S}{2}T[0,2] \\ -\frac{S}{2}T[1,1] + (1+S)T[1,2] - \frac{S}{2}T[1,3] &= \frac{S}{2}T[0,1] + (1-S)T[0,2] + \frac{S}{2}T[0,3] \\ -\frac{S}{2}T[1,2] + (1+S)T[1,3] - \frac{S}{2}T[1,4] &= \frac{S}{2}T[0,2] + (1-S)T[0,3] + \frac{S}{2}T[0,4] \end{aligned}$$

Reordenando

$$\begin{aligned} (1+S)T[1,1] - \frac{S}{2}T[1,2] + 0T[1,3] &= (1-S)T[0,1] + \frac{S}{2}T[0,2] + 0T[0,3] + \frac{S}{2}T[0,0] + \frac{S}{2}T[1,0] \\ -\frac{S}{2}T[1,1] + (1+S)T[1,2] - \frac{S}{2}T[1,3] &= \frac{S}{2}T[0,1] + (1-S)T[0,2] + \frac{S}{2}T[0,3] \\ 0T[1,1] - \frac{S}{2}T[1,2] + (1+S)T[1,3] &= 0T[0,1] + \frac{S}{2}T[0,2] + (1-S)T[0,3] + \frac{S}{2}T[0,4] + \frac{S}{2}T[1,4] \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1+S & -\frac{S}{2} & 0 \\ -\frac{S}{2} & 1+S & -\frac{S}{2} \\ 0 & -\frac{S}{2} & 1+S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} 1-S & \frac{S}{2} & 0 \\ \frac{S}{2} & 1-S & \frac{S}{2} \\ 0 & \frac{S}{2} & 1-S \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[0,0] \\ 0 \\ T[0,4] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[1,0] \\ 0 \\ T[1,4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1,x]$, también puede escribirse como

$$\begin{bmatrix} 1+S & -\frac{S}{2} & 0 \\ -\frac{S}{2} & 1+S & -\frac{S}{2} \\ 0 & -\frac{S}{2} & 1+S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} 1-S & \frac{S}{2} & 0 \\ \frac{S}{2} & 1-S & \frac{S}{2} \\ 0 & \frac{S}{2} & 1-S \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[0,0] + T[1,0] \\ 0 \\ T[0,4] + T[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{aligned} &\begin{bmatrix} 1+S & -\frac{S}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{S}{2} & 1+S & -\frac{S}{2} & \ddots & & & \vdots \\ 0 & -\frac{S}{2} & 1+S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1+S & -\frac{S}{2} & 0 \\ \vdots & & & \ddots & -\frac{S}{2} & 1+S & -\frac{S}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & -\frac{S}{2} & 1+S \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} \\ &= \begin{bmatrix} 1-S & \frac{S}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{S}{2} & 1-S & \frac{S}{2} & \ddots & & & \vdots \\ 0 & \frac{S}{2} & 1-S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1-S & \frac{S}{2} & 0 \\ \vdots & & & \ddots & \frac{S}{2} & 1-S & \frac{S}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{S}{2} & 1-S \end{bmatrix} \begin{bmatrix} T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[n,0] + T[n+1,0] \\ 0 \\ \vdots \\ \vdots \\ 0 \\ T[n,s+1] + T[n+1,s+1] \end{bmatrix} \end{aligned} \tag{5.9}$$

5.2.4. θ

Reemplazando en el método de Crank-Nicolson, $\theta = \frac{1}{2}$

$$\theta ST(n, j+1) + (1 - 2\theta S)T(n, j) + \theta ST(n, j-1) = -\theta ST(n+1, j+1) + (1 + 2\theta S)T(n+1, j) - \theta ST(n+1, j-1)$$

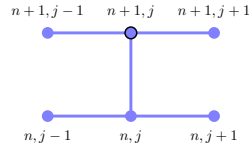


Figura 5.33: Esquema θ

Realizando los mismos pasos para los métodos anteriores, se obtiene

$$\begin{bmatrix} 1 + 2\theta S & -\theta S & 0 \\ -\theta S & 1 + 2\theta S & -\theta S \\ 0 & -\theta S & 1 + 2\theta S \end{bmatrix} \begin{bmatrix} T[1, 1] \\ T[1, 2] \\ T[1, 3] \end{bmatrix} = \begin{bmatrix} 1 - 2\theta S & \theta S & 0 \\ \theta S & 1 - 2\theta S & \theta S \\ 0 & \theta S & 1 - 2\theta S \end{bmatrix} \begin{bmatrix} T[0, 1] \\ T[0, 2] \\ T[0, 3] \end{bmatrix} + \theta S \begin{bmatrix} T[0, 0] + T[1, 0] \\ 0 \\ T[0, 4] + T[1, 4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{aligned} & \begin{bmatrix} 1 + 2\theta S & -\theta S & 0 & \cdots & \cdots & \cdots & 0 \\ -\theta S & 1 + 2\theta S & -\theta S & \ddots & & & \vdots \\ 0 & -\theta S & 1 + 2\theta S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 + 2\theta S & -\theta S & 0 \\ \vdots & & & \ddots & -\theta S & 1 + 2\theta S & -\theta S \\ 0 & \cdots & \cdots & \cdots & 0 & -\theta S & 1 + 2\theta S \end{bmatrix} \begin{bmatrix} T[n+1, 1] \\ T[n+1, 2] \\ \vdots \\ \vdots \\ T[n+1, s-1] \\ T[n+1, s] \end{bmatrix} \\ & = \begin{bmatrix} 1 - 2\theta S & \theta S & 0 & \cdots & \cdots & \cdots & 0 \\ \theta S & 1 - 2\theta S & \theta S & \ddots & & & \vdots \\ 0 & \theta S & 1 - 2\theta S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 - 2\theta S & \theta S & 0 \\ \vdots & & & \ddots & \theta S & 1 - 2\theta S & \theta S \\ 0 & \cdots & \cdots & \cdots & 0 & \theta S & 1 - 2\theta S \end{bmatrix} \begin{bmatrix} T[n, 1] \\ T[n, 2] \\ \vdots \\ \vdots \\ T[n, s-1] \\ T[n, s] \end{bmatrix} \\ & + \theta S \begin{bmatrix} T[n, 0] + T[n+1, 0] \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ T[n, s+1] + T[n+1, s+1] \end{bmatrix} \end{aligned} \quad (5.10)$$

Capítulo 6

Aplicaciones en hidráulica

6.1. Advección-Difusión

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = D \frac{\partial^2 T}{\partial x^2}$$

6.1.1. FTCS

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} + u \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x} = D \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reordenando

$$T(t + \Delta t) - T(t) + \frac{u\Delta t}{2\Delta x} [T(x + \Delta x) - T(x - \Delta x)] = \frac{D\Delta t}{\Delta x^2} [T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reemplazando por C y S

$$T(t + \Delta t) - T(t) + \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)] = S [T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reordenando

$$T(t + \Delta t) = T(t) - \frac{C}{2} [T(x + \Delta x) - T(x - \Delta x)] + S [T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t + \Delta t, x) = T(t, x) - \frac{C}{2} [T(t, x + \Delta x) - T(t, x - \Delta x)] + S [T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) - \frac{C}{2} [T(n, j + 1) - T(n, j - 1)] + S [T(n, j + 1) - 2T(n, j) + T(n, j - 1)]$$

Reordenando

$$T(n + 1, j) = \left(S + \frac{C}{2} \right) T(n, j - 1) + (1 - 2S) T(n, j) + \left(S - \frac{C}{2} \right) T(n, j + 1) \quad (6.1)$$

6.1.2. FTCS implícito

Aproximación hacia atrás de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t) - T(t - \Delta t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t) - T(t - \Delta t)}{\Delta t} + u \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x} = D \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reordenando

$$T(t) - T(t - \Delta t) + \frac{u\Delta t}{2\Delta x}[T(x + \Delta x) - T(x - \Delta x)] = \frac{D\Delta t}{\Delta x^2}[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reemplazando por C y S

$$T(t) - T(t - \Delta t) + \frac{C}{2}[T(x + \Delta x) - T(x - \Delta x)] = S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reordenando

$$T(t) = T(t - \Delta t) - \frac{C}{2}[T(x + \Delta x) - T(x - \Delta x)] + S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t, x) = T(t - \Delta t, x) - \frac{C}{2}[T(t, x + \Delta x) - T(t, x - \Delta x)] + S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Reemplazando $t = t + \Delta t$

$$T(t + \Delta t, x) = T(t, x) - \frac{C}{2}[T(t + \Delta t, x + \Delta x) - T(t + \Delta t, x - \Delta x)] + S[T(t + \Delta t, x + \Delta x) - 2T(t + \Delta t, x) + T(t + \Delta t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) - \frac{C}{2}[T(n + 1, j + 1) - T(n + 1, j - 1)] + S[T(n + 1, j + 1) - 2T(n + 1, j) + T(n + 1, j - 1)]$$

Reordenando

$$T(n, j) = -\left(S + \frac{C}{2}\right)T(n + 1, j - 1) + (1 + 2S)T(n + 1, j) - \left(S - \frac{C}{2}\right)T(n + 1, j + 1) \quad (6.2)$$

Para generalizar el método se usará tres pasos en el espacio

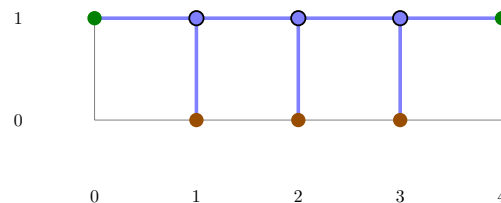


Figura 6.1: Movimiento del esquema para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned} -\left(S + \frac{C}{2}\right)T[1, 0] + (1 + 2S)T[1, 1] - \left(S - \frac{C}{2}\right)T[1, 2] &= T[0, 1] \\ -\left(S + \frac{C}{2}\right)T[1, 1] + (1 + 2S)T[1, 2] - \left(S - \frac{C}{2}\right)T[1, 3] &= T[0, 2] \\ -\left(S + \frac{C}{2}\right)T[1, 2] + (1 + 2S)T[1, 3] - \left(S - \frac{C}{2}\right)T[1, 4] &= T[0, 3] \end{aligned}$$

Reescribiendo

$$\begin{aligned} -\left(S + \frac{C}{2}\right)T[1, 0] + (1 + 2S)T[1, 1] - \left(S - \frac{C}{2}\right)T[1, 2] + 0T[1, 3] + 0T[1, 4] &= T[0, 1] \\ 0T[1, 0] - \left(S + \frac{C}{2}\right)T[1, 1] + (1 + 2S)T[1, 2] - \left(S - \frac{C}{2}\right)T[1, 3] + 0T[1, 4] &= T[0, 2] \\ 0T[1, 0] + 0T[1, 1] - \left(S + \frac{C}{2}\right)T[1, 2] + (1 + 2S)T[1, 3] - \left(S - \frac{C}{2}\right)T[1, 4] &= T[0, 3] \end{aligned}$$

Reordenando

$$\begin{aligned} (1 + 2S)T[1, 1] - \left(S - \frac{C}{2}\right)T[1, 2] + 0T[1, 3] &= T[0, 1] + \left(S + \frac{C}{2}\right)T[1, 0] + 0T[1, 0] \\ -\left(S + \frac{C}{2}\right)T[1, 1] + (1 + 2S)T[1, 2] - \left(S - \frac{C}{2}\right)T[1, 3] &= T[0, 2] + 0T[1, 0] + 0T[1, 4] \\ 0T[1, 1] - \left(S + \frac{C}{2}\right)T[1, 2] + (1 + 2S)T[1, 3] &= T[0, 3] + 0T[1, 0] + \left(S - \frac{C}{2}\right)T[1, 4] \end{aligned}$$

Simplificando

$$\begin{aligned} (1 + 2S)T[1, 1] - \left(S - \frac{C}{2}\right)T[1, 2] + 0T[1, 3] &= T[0, 1] + \left(S + \frac{C}{2}\right)T[1, 0] \\ -\left(S + \frac{C}{2}\right)T[1, 1] + (1 + 2S)T[1, 2] - \left(S - \frac{C}{2}\right)T[1, 3] &= T[0, 2] \\ 0T[1, 1] - \left(S + \frac{C}{2}\right)T[1, 2] + (1 + 2S)T[1, 3] &= T[0, 3] + \left(S - \frac{C}{2}\right)T[1, 4] \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1 + 2S & -\left(S - \frac{C}{2}\right) & 0 \\ -\left(S + \frac{C}{2}\right) & 1 + 2S & -\left(S - \frac{C}{2}\right) \\ 0 & -\left(S + \frac{C}{2}\right) & 1 + 2S \end{bmatrix} \begin{bmatrix} T[1, 1] \\ T[1, 2] \\ T[1, 3] \end{bmatrix} = \begin{bmatrix} T[0, 1] \\ T[0, 2] \\ T[0, 3] \end{bmatrix} + S \begin{bmatrix} T[1, 0] \\ 0 \\ T[1, 4] \end{bmatrix} + \frac{C}{2} \begin{bmatrix} T[1, 0] \\ 0 \\ -T[1, 4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1, x]$, también puede escribirse como

$$\begin{bmatrix} 1 + 2S & -\left(S - \frac{C}{2}\right) & 0 \\ -\left(S + \frac{C}{2}\right) & 1 + 2S & -\left(S - \frac{C}{2}\right) \\ 0 & -\left(S + \frac{C}{2}\right) & 1 + 2S \end{bmatrix} \begin{bmatrix} T[1, 1] \\ T[1, 2] \\ T[1, 3] \end{bmatrix} = \begin{bmatrix} T[0, 1] + \left(S + \frac{C}{2}\right)T[1, 0] \\ T[0, 2] \\ T[0, 3] + \left(S - \frac{C}{2}\right)T[1, 4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{aligned}
 & \begin{bmatrix} 1+2S & -\left(S-\frac{C}{2}\right) & 0 & \cdots & \cdots & \cdots & 0 \\ -\left(S+\frac{C}{2}\right) & 1+2S & -\left(S-\frac{C}{2}\right) & \ddots & & & \vdots \\ 0 & -\left(S+\frac{C}{2}\right) & 1+2S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1+2S & -\left(S-\frac{C}{2}\right) & 0 \\ \vdots & & & \ddots & -\left(S+\frac{C}{2}\right) & 1+2S & -\left(S-\frac{C}{2}\right) \\ 0 & \cdots & \cdots & \cdots & 0 & -\left(S+\frac{C}{2}\right) & 1+2S \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} \\
 & = \begin{bmatrix} T[n,1] + \left(S+\frac{C}{2}\right)T[n+1,0] \\ T[n,2] \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] + \left(S-\frac{C}{2}\right)T[n+1,s+1] \end{bmatrix} \quad (6.3)
 \end{aligned}$$

6.1.3. Crank-Nicolson

Puede derivarse de métodos FTCS

$$T(n+1, j) = \left(S + \frac{C}{2}\right)T(n, j-1) + (1-2S)T(n, j) + \left(S - \frac{C}{2}\right)T(n, j+1) \quad \text{explícito}$$

$$T(n, j) = -\left(S + \frac{C}{2}\right)T(n+1, j-1) + (1+2S)T(n+1, j) - \left(S - \frac{C}{2}\right)T(n+1, j+1) \quad \text{implícito}$$

Reordenando

$$T(n+1, j) - T(n, j) = \left(S + \frac{C}{2}\right)T(n, j-1) - 2ST(n, j) + \left(S - \frac{C}{2}\right)T(n, j+1)$$

$$T(n+1, j) - T(n, j) = \left(S + \frac{C}{2}\right)T(n+1, j-1) - 2ST(n+1, j) + \left(S - \frac{C}{2}\right)T(n+1, j+1)$$

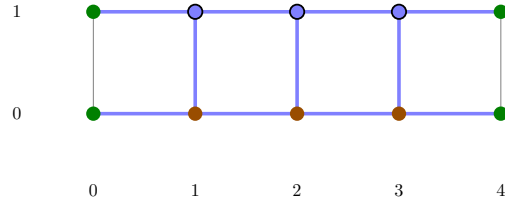
Multiplicando por $\frac{1}{2}$ y sumando

$$\begin{aligned}
 T(n+1, j) - T(n, j) &= \left(\frac{S}{2} + \frac{C}{4}\right)T(n, j-1) - ST(n, j) + \left(\frac{S}{2} - \frac{C}{4}\right)T(n, j+1) \\
 &+ \left(\frac{S}{2} + \frac{C}{4}\right)T(n+1, j-1) - ST(n+1, j) + \left(\frac{S}{2} - \frac{C}{4}\right)T(n+1, j+1)
 \end{aligned}$$

Reordenando

$$\begin{aligned}
 & \left(\frac{S}{2} + \frac{C}{4}\right)T(n, j-1) + (1-S)T(n, j) + \left(\frac{S}{2} - \frac{C}{4}\right)T(n, j+1) \\
 &= -\left(\frac{S}{2} + \frac{C}{4}\right)T(n+1, j-1) + (1+S)T(n+1, j) - \left(\frac{S}{2} - \frac{C}{4}\right)T(n+1, j+1)
 \end{aligned}$$

Para generalizar el método se usará tres pasos en el espacio

Figura 6.2: Movimiento del esquema para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned}
 -\left(\frac{S}{2} + \frac{C}{4}\right)T[1,0] + (1+S)T[1,1] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,2] &= \left(\frac{S}{2} + \frac{C}{4}\right)T[0,0] + (1-S)T[0,1] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,2] \\
 -\left(\frac{S}{2} + \frac{C}{4}\right)T[1,1] + (1+S)T[1,2] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,3] &= \left(\frac{S}{2} + \frac{C}{4}\right)T[0,1] + (1-S)T[0,2] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,3] \\
 -\left(\frac{S}{2} + \frac{C}{4}\right)T[1,2] + (1+S)T[1,3] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,4] &= \left(\frac{S}{2} + \frac{C}{4}\right)T[0,2] + (1-S)T[0,3] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,4]
 \end{aligned}$$

Reordenando

$$\begin{aligned}
 (1+S)T[1,1] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,2] + \quad \quad \quad 0T[1,3] &= \quad (1-S)T[0,1] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,2] + \quad \quad \quad 0T[0,3] + \left(\frac{S}{2} + \frac{C}{4}\right)T[0,0] + \left(\frac{S}{2} + \frac{C}{4}\right)T[1,0] \\
 -\left(\frac{S}{2} + \frac{C}{4}\right)T[1,1] + \quad (1+S)T[1,2] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,3] &= \left(\frac{S}{2} + \frac{C}{4}\right)T[0,1] + \quad (1-S)T[0,2] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,3] \\
 0T[1,1] - \left(\frac{S}{2} + \frac{C}{4}\right)T[1,2] + \quad (1+S)T[1,3] &= \quad 0T[0,1] + \left(\frac{S}{2} + \frac{C}{4}\right)T[0,2] + \quad (1-S)T[0,3] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,4] + \left(\frac{S}{2} - \frac{C}{4}\right)T[1,4]
 \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1+S & -\left(\frac{S}{2} - \frac{C}{4}\right) & 0 \\ -\left(\frac{S}{2} + \frac{C}{4}\right) & 1+S & -\left(\frac{S}{2} - \frac{C}{4}\right) \\ 0 & -\left(\frac{S}{2} + \frac{C}{4}\right) & 1+S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} 1-S & \frac{S}{2} - \frac{C}{4} & 0 \\ \frac{S}{2} + \frac{C}{4} & 1-S & \frac{S}{2} - \frac{C}{4} \\ 0 & \frac{S}{2} + \frac{C}{4} & 1-S \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \begin{bmatrix} \left(\frac{S}{2} + \frac{C}{4}\right)(T[0,0] + T[1,0]) \\ 0 \\ \left(\frac{S}{2} - \frac{C}{4}\right)(T[0,4] + T[1,4]) \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{aligned}
 & \begin{bmatrix} 1+S & -\left(\frac{S}{2}-\frac{C}{4}\right) & 0 & \cdots & \cdots & \cdots & 0 \\ -\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S & -\left(\frac{S}{2}-\frac{C}{4}\right) & \ddots & & & \vdots \\ 0 & -\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1+S & -\left(\frac{S}{2}-\frac{C}{4}\right) & 0 \\ \vdots & & & \ddots & -\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S & -\left(\frac{S}{2}-\frac{C}{4}\right) \\ 0 & \cdots & \cdots & \cdots & 0 & -\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} \\
 = & \begin{bmatrix} 1-S & \frac{S}{2}-\frac{C}{4} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{S}{2}+\frac{C}{4} & 1-S & \frac{S}{2}-\frac{C}{4} & \ddots & & & \vdots \\ 0 & \frac{S}{2}+\frac{C}{4} & 1-S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1-S & \frac{S}{2}-\frac{C}{4} & 0 \\ \vdots & & & \ddots & \frac{S}{2}+\frac{C}{4} & 1-S & \frac{S}{2}-\frac{C}{4} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{S}{2}+\frac{C}{4} & 1-S \end{bmatrix} \begin{bmatrix} T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] \end{bmatrix} \\
 + & \begin{bmatrix} \left(\frac{S}{2}+\frac{C}{4}\right)(T[n,0]+T[n+1,0]) \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \left(\frac{S}{2}-\frac{C}{4}\right)(T[n,s+1]+T[n+1,s+1]) \end{bmatrix}
 \end{aligned} \tag{6.4}$$