Funciones de forma Lagrangianas
Generación de funciones usando producto tensorial

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Resumen

El presente trabajo describe los pasos para generar funciones de forma bidimensionales y tridimensionales mediante producto matricial y tensorial.

Notación

- $l^{(1)}$ Función de forma unidimensional Lagrangiana
- $l^{(2)}$ Función de forma bidimensional Lagrangiana
- $l^{(3)}$ Función de forma trididimensional Lagrangiana

1. Funciones de forma unidimensionales

1.1. Definición

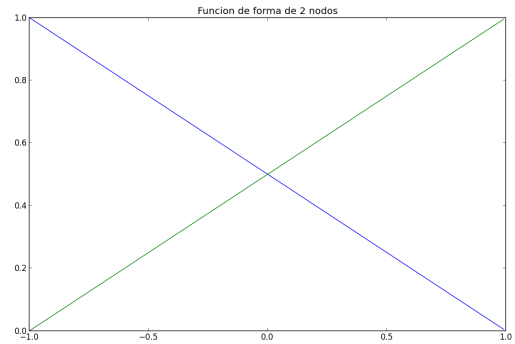
Las funciones de forma Lagrangianas [1] son polinomios con dominio $-1 \leqslant x \leqslant 1$, la fórmula para generarlos es la siguiente:

$$l_j^{(n)} = \prod_{i=0, i\neq j}^{k} \frac{x - x_i}{x_j - x_i}$$

1.2. Ejemplo $l^{(1)}$

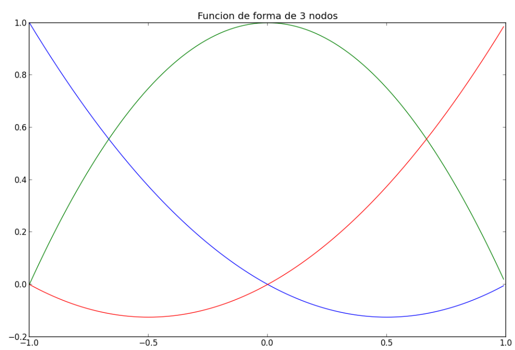
Función de forma Lagrangiana unidimensional de 2 nodos:

$$\begin{aligned} l_1^{(1)} &= -\frac{1}{2}\xi + \frac{1}{2} \\ l_2^{(1)} &= \frac{1}{2}\xi + \frac{1}{2} \end{aligned}$$



Función de forma Lagrangiana unidimensional de 3 nodos:

$$
\begin{aligned}
l_1^{(1)} &= \frac{1}{2}\xi^2 - \frac{1}{2}\xi \\
l_2^{(1)} &= -\xi^2 + 1 \\
l_3^{(1)} &= \frac{1}{2}\xi^2 + \frac{1}{2}\xi
\end{aligned}
$$



2. Funciones de forma bidimensionales

2.1. Método

Las $l^{(2)}$ se generar a partir de $l^{(1)}$, multiplicando matrices:

$$\left[l^{(1)}\right]\left[l^{(1)}\right]^T = \left[l^{(2)}\right]$$

En donde $\left[l^{(1)}\right]$ es una matriz columna con j-elementos, $\left[l^{(1)}\right]^T$ es una matriz fila con j-elementos y $\left[l^{(2)}\right]$ es una matriz cuadrada.
Si recorremos los elementos que forman parte de la matriz $\left[l^{(2)}\right]$, uno a uno desde $l_1^{(2)}$ hasta $l_n^{(2)}$ forman una espiral, si graficamos las funciones que forman la matriz en el orden anterior observamos que el punto (ξ, η) en el que la función vale 1 también hace un recorrido en espiral.

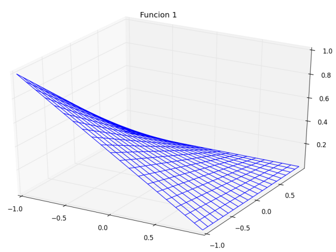
2.2. Ejemplo $l^{(2)}$

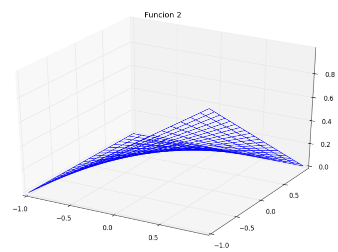
Elemento rectangular de 4 nodos:

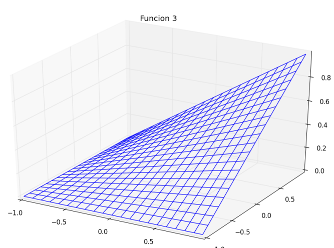
$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\xi + \frac{1}{2} \\ \frac{1}{2}\xi + \frac{1}{2} \end{bmatrix}$$

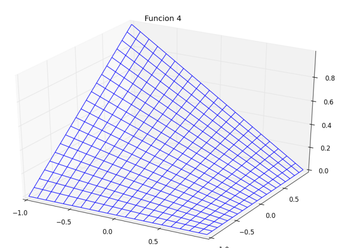
$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \end{bmatrix} \begin{bmatrix} l_1^{(1)} & l_2^{(1)} \end{bmatrix} = \begin{bmatrix} l_1^{(2)} & l_4^{(2)} \\ l_2^{(2)} & l_3^{(2)} \end{bmatrix}$$

$$\begin{aligned} l_1^{(2)} &= \frac{1}{4}\xi^2\eta^2 - \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2 + \frac{1}{4}\xi\eta \\ l_2^{(2)} &= -\frac{1}{2}\xi^2\eta^2 + \frac{1}{2}\xi^2\eta + \frac{1}{2}\eta^2 - \frac{1}{2}\eta \\ l_3^{(2)} &= \frac{1}{4}\xi^2\eta^2 - \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2 - \frac{1}{4}\xi\eta \\ l_4^{(2)} &= -\frac{1}{2}\xi^2\eta^2 + \frac{1}{2}\xi^2 - \frac{1}{2}\xi\eta^2 + \frac{1}{2}\xi \end{aligned}$$







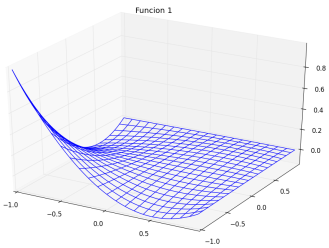


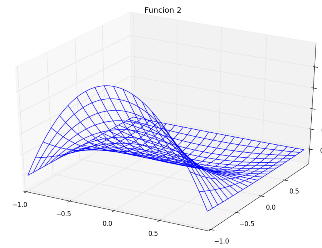
Elemento rectangular de 9 nodos:

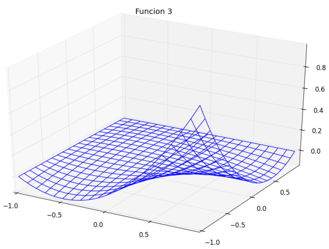
$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \\ l_3^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\xi^2 - \frac{1}{2}\xi \\ -\xi^2 + 1 \\ \frac{1}{2}\xi^2 + \frac{1}{2}\xi \end{bmatrix}$$

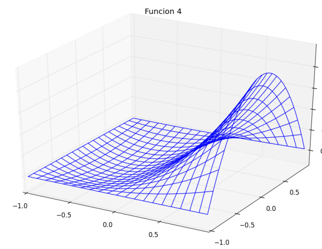
$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \\ l_3^{(1)} \end{bmatrix} \begin{bmatrix} l_1^{(1)} & l_2^{(1)} & l_3^{(1)} \end{bmatrix} = \begin{bmatrix} l_1^{(2)} & l_8^{(2)} & l_7^{(2)} \\ l_2^{(2)} & l_9^{(2)} & l_6^{(2)} \\ l_3^{(2)} & l_4^{(2)} & l_5^{(2)} \end{bmatrix}$$

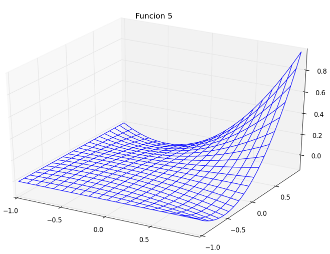
$$
\begin{aligned}
l_1^{(2)} &= \frac{1}{4}\xi^2\eta^2 - \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2 + \frac{1}{4}\xi\eta \\
l_2^{(2)} &= -\frac{1}{2}\xi^2\eta^2 + \frac{1}{2}\xi^2\eta + \frac{1}{2}\eta^2 - \frac{1}{2}\eta \\
l_3^{(2)} &= \frac{1}{4}\xi^2\eta^2 - \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2 - \frac{1}{4}\xi\eta \\
l_4^{(2)} &= -\frac{1}{2}\xi^2\eta^2 + \frac{1}{2}\xi^2 - \frac{1}{2}\xi\eta^2 + \frac{1}{2}\xi \\
l_5^{(2)} &= \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2 + \frac{1}{4}\xi\eta \\
l_6^{(2)} &= -\frac{1}{2}\xi^2\eta^2 - \frac{1}{2}\xi^2\eta + \frac{1}{2}\eta^2 + \frac{1}{2}\eta \\
l_7^{(2)} &= \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2 - \frac{1}{4}\xi\eta \\
l_8^{(2)} &= -\frac{1}{2}\xi^2\eta^2 + \frac{1}{2}\xi^2 + \frac{1}{2}\xi\eta^2 - \frac{1}{2}\xi \\
l_9^{(2)} &= \xi^2\eta^2 - \xi^2 - \eta^2 + 1
\end{aligned}
$$

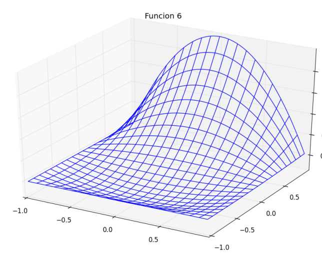

Funcion 1

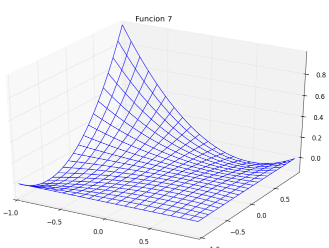

Funcion 2

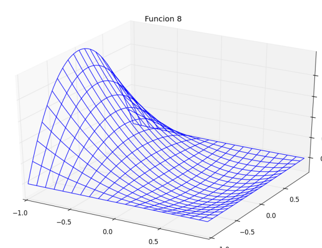

Funcion 3

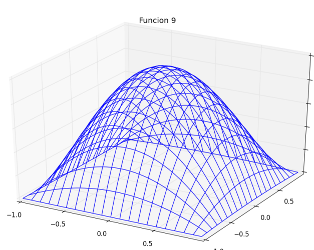

Funcion 4


Funcion 5


Funcion 6


Funcion 7


Funcion 8


Funcion 9

3. Funciones de forma tridimensionales

3.1. Método

Las $l^{(3)}$ también se generar a partir de $l^{(1)}$, realizando un producto tensorial [2].

$$\left[l^{(1)}\right]\left[l^{(1)}\right]^T\left[l^{(1)}\right]^T=\left[l^{(3)}\right]$$

En donde $\left[l^{(1)}\right]$ es una matriz columna con j-elementos, $\left[l^{(1)}\right]^T$ es una matriz fila con j-elementos y $\left[l^{(3)}\right]$ es una hipermatriz.

3.2. Ejemplo $l^{(3)}$

Elemento hexaédrico de 8 nodos:

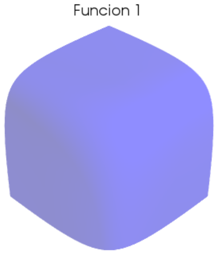
$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\xi+\frac{1}{2} \\ \frac{1}{2}\xi+\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \end{bmatrix}\begin{bmatrix} l_1^{(1)} & l_2^{(1)} \end{bmatrix}\begin{bmatrix} l_1^{(1)} & l_2^{(1)} \end{bmatrix} = \begin{bmatrix}\begin{bmatrix} l_1^{(3)} & l_5^{(3)} \\ l_2^{(3)} & l_6^{(3)} \end{bmatrix} & \begin{bmatrix} l_4^{(3)} & l_8^{(3)} \\ l_3^{(3)} & l_7^{(3)} \end{bmatrix}\end{bmatrix}$$

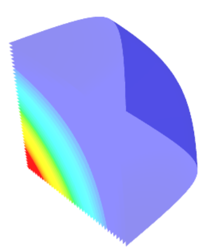
$$\begin{aligned}
l_1^{(3)} &= -\frac{1}{8}\xi\eta\zeta+\frac{1}{8}\xi\eta+\frac{1}{8}\xi\zeta-\frac{1}{8}\xi+\frac{1}{8}\eta\zeta-\frac{1}{8}\eta-\frac{1}{8}\zeta+\frac{1}{8} \\
l_2^{(3)} &= \frac{1}{8}\xi\eta\zeta-\frac{1}{8}\xi\eta-\frac{1}{8}\xi\zeta+\frac{1}{8}\xi+\frac{1}{8}\eta\zeta-\frac{1}{8}\eta-\frac{1}{8}\zeta+\frac{1}{8} \\
l_6^{(3)} &= -\frac{1}{8}\xi\eta\zeta-\frac{1}{8}\xi\eta+\frac{1}{8}\xi\zeta+\frac{1}{8}\xi-\frac{1}{8}\eta\zeta-\frac{1}{8}\eta+\frac{1}{8}\zeta+\frac{1}{8} \\
l_5^{(3)} &= \frac{1}{8}\xi\eta\zeta+\frac{1}{8}\xi\eta-\frac{1}{8}\xi\zeta-\frac{1}{8}\xi-\frac{1}{8}\eta\zeta-\frac{1}{8}\eta+\frac{1}{8}\zeta+\frac{1}{8} \\
l_4^{(3)} &= \frac{1}{8}\xi\eta\zeta-\frac{1}{8}\xi\eta+\frac{1}{8}\xi\zeta-\frac{1}{8}\xi-\frac{1}{8}\eta\zeta+\frac{1}{8}\eta-\frac{1}{8}\zeta+\frac{1}{8} \\
l_3^{(3)} &= -\frac{1}{8}\xi\eta\zeta+\frac{1}{8}\xi\eta-\frac{1}{8}\xi\zeta+\frac{1}{8}\xi-\frac{1}{8}\eta\zeta+\frac{1}{8}\eta-\frac{1}{8}\zeta+\frac{1}{8}
\end{aligned}$$

$$\begin{aligned}
l_7^{(3)} &= \frac{1}{8}\xi\eta\zeta+\frac{1}{8}\xi\eta+\frac{1}{8}\xi\zeta+\frac{1}{8}\xi+\frac{1}{8}\eta\zeta+\frac{1}{8}\eta+\frac{1}{8}\zeta+\frac{1}{8} \\
l_8^{(3)} &= -\frac{1}{8}\xi\eta\zeta-\frac{1}{8}\xi\eta-\frac{1}{8}\xi\zeta-\frac{1}{8}\xi+\frac{1}{8}\eta\zeta+\frac{1}{8}\eta+\frac{1}{8}\zeta+\frac{1}{8}
\end{aligned}$$

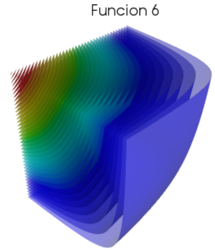
Funcion 1

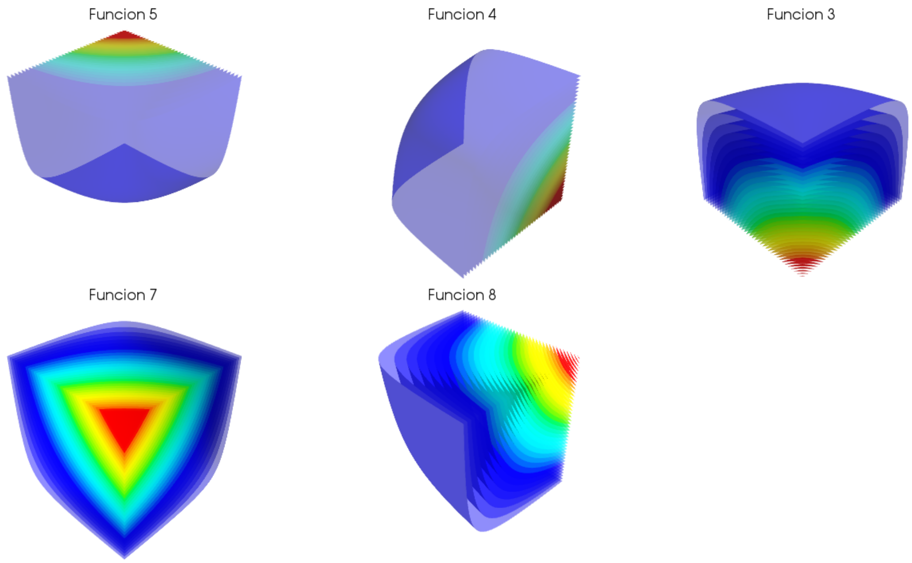


Funcion 2



Funcion 6





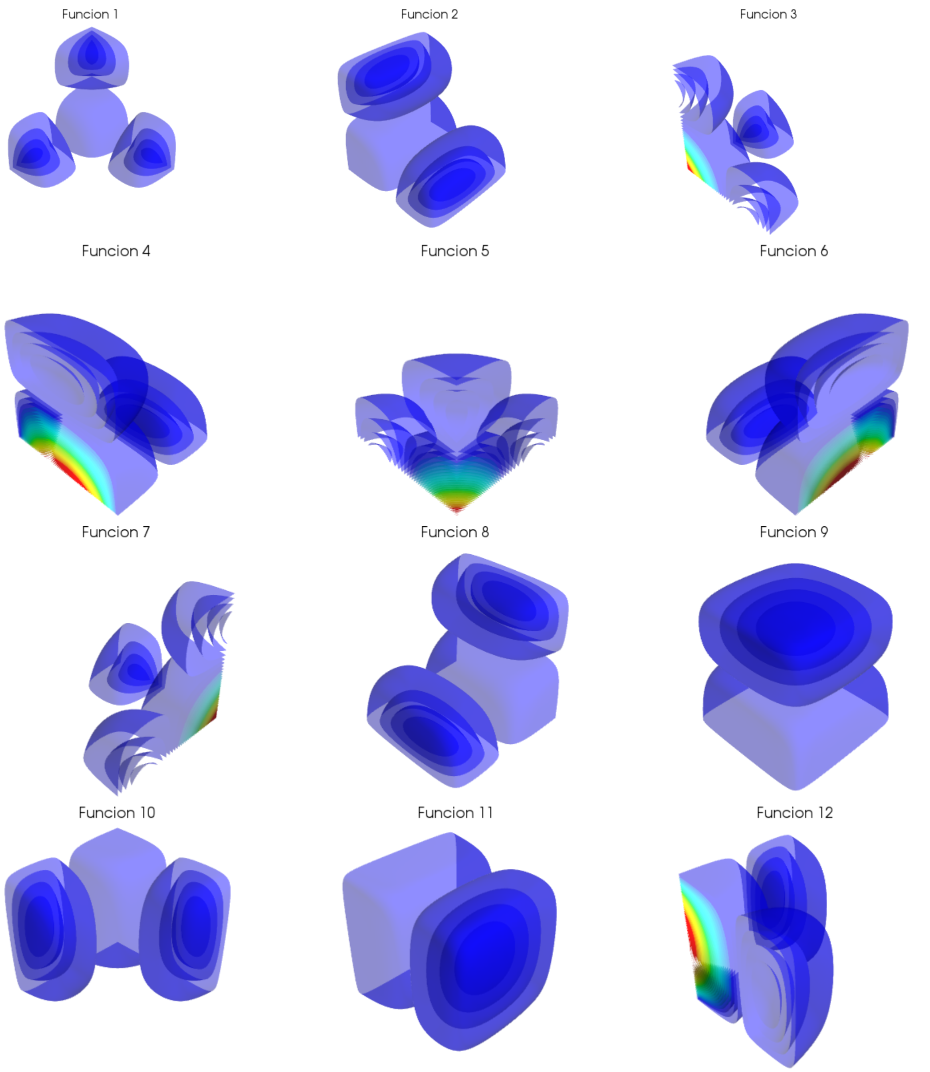
Elemento hexaédrico de 27 nodos:

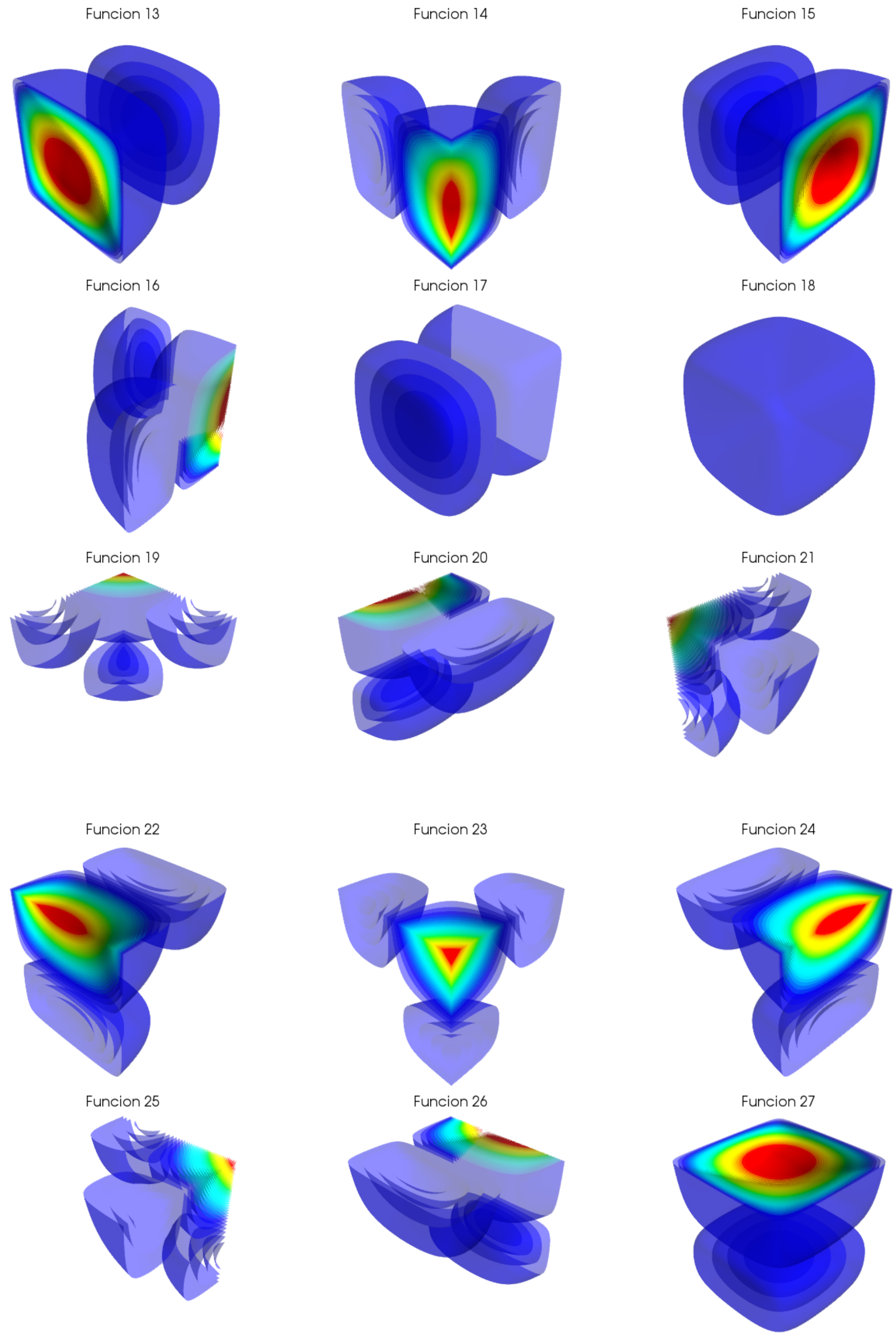
$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \\ l_3^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\xi^2 - \frac{1}{2}\xi \\ -\xi^2 + 1 \\ \frac{1}{2}\xi^2 + \frac{1}{2}\xi \end{bmatrix}$$

$$\begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \\ l_3^{(1)} \end{bmatrix} \begin{bmatrix} l_1^{(1)} & l_2^{(1)} & l_3^{(1)} \end{bmatrix} \begin{bmatrix} l_1^{(1)} & l_2^{(1)} & l_3^{(1)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} l_1^{(3)} & l_{10}^{(3)} & l_{19}^{(3)} \\ l_2^{(3)} & l_{11}^{(3)} & l_{20}^{(3)} \\ l_3^{(3)} & l_{12}^{(3)} & l_{21}^{(3)} \end{bmatrix} \begin{bmatrix} l_8^{(3)} & l_{17}^{(3)} & l_{26}^{(3)} \\ l_9^{(3)} & l_{18}^{(3)} & l_{27}^{(3)} \\ l_4^{(3)} & l_{13}^{(3)} & l_{22}^{(3)} \end{bmatrix} \begin{bmatrix} l_7^{(3)} & l_{16}^{(3)} & l_{25}^{(3)} \\ l_6^{(3)} & l_{15}^{(3)} & l_{24}^{(3)} \\ l_5^{(3)} & l_{14}^{(3)} & l_{23}^{(3)} \end{bmatrix} \end{bmatrix}$$

$$
\begin{aligned}
l_1^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 - \frac{1}{8}\xi^2\eta^2\zeta - \frac{1}{8}\xi^2\eta\zeta^2 + \frac{1}{8}\xi^2\eta\zeta - \frac{1}{8}\xi\eta^2\zeta^2 + \frac{1}{8}\xi\eta^2\zeta + \frac{1}{8}\xi\eta\zeta^2 - \frac{1}{8}\xi\eta\zeta \\
l_2^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2\zeta + \frac{1}{4}\xi^2\eta\zeta^2 - \frac{1}{4}\xi^2\eta\zeta + \frac{1}{4}\eta^2\zeta^2 - \frac{1}{4}\eta^2\zeta - \frac{1}{4}\eta\zeta^2 + \frac{1}{4}\eta\zeta \\
l_3^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 - \frac{1}{8}\xi^2\eta^2\zeta - \frac{1}{8}\xi^2\eta\zeta^2 + \frac{1}{8}\xi^2\eta\zeta + \frac{1}{8}\xi\eta^2\zeta^2 - \frac{1}{8}\xi\eta^2\zeta - \frac{1}{8}\xi\eta\zeta^2 + \frac{1}{8}\xi\eta\zeta \\
l_{12}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta\zeta^2 - \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2\zeta^2 + \frac{1}{4}\xi\eta^2 + \frac{1}{4}\xi\eta\zeta^2 - \frac{1}{4}\xi\eta \\
l_{21}^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 + \frac{1}{8}\xi^2\eta^2\zeta - \frac{1}{8}\xi^2\eta\zeta^2 - \frac{1}{8}\xi^2\eta\zeta + \frac{1}{8}\xi\eta^2\zeta^2 + \frac{1}{8}\xi\eta^2\zeta - \frac{1}{8}\xi\eta\zeta^2 - \frac{1}{8}\xi\eta\zeta \\
l_{20}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 - \frac{1}{4}\xi^2\eta^2\zeta + \frac{1}{4}\xi^2\eta\zeta^2 + \frac{1}{4}\xi^2\eta\zeta + \frac{1}{4}\eta^2\zeta^2 + \frac{1}{4}\eta^2\zeta - \frac{1}{4}\eta\zeta^2 - \frac{1}{4}\eta\zeta \\
l_{19}^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 + \frac{1}{8}\xi^2\eta^2\zeta - \frac{1}{8}\xi^2\eta\zeta^2 - \frac{1}{8}\xi^2\eta\zeta - \frac{1}{8}\xi\eta^2\zeta^2 - \frac{1}{8}\xi\eta^2\zeta + \frac{1}{8}\xi\eta\zeta^2 + \frac{1}{8}\xi\eta\zeta \\
l_{10}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta\zeta^2 - \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2\zeta^2 - \frac{1}{4}\xi\eta^2 - \frac{1}{4}\xi\eta\zeta^2 + \frac{1}{4}\xi\eta \\
l_{11}^{(3)} &= \frac{1}{2}\xi^2\eta^2\zeta^2 - \frac{1}{2}\xi^2\eta^2 - \frac{1}{2}\xi^2\eta\zeta^2 + \frac{1}{2}\xi^2\eta - \frac{1}{2}\eta^2\zeta^2 + \frac{1}{2}\eta^2 + \frac{1}{2}\eta\zeta^2 - \frac{1}{2}\eta \\
l_8^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2\zeta + \frac{1}{4}\xi^2\zeta^2 - \frac{1}{4}\xi^2\zeta + \frac{1}{4}\xi\eta^2\zeta^2 - \frac{1}{4}\xi\eta^2\zeta - \frac{1}{4}\xi\zeta^2 + \frac{1}{4}\xi\zeta \\
l_9^{(3)} &= \frac{1}{2}\xi^2\eta^2\zeta^2 - \frac{1}{2}\xi^2\eta^2\zeta - \frac{1}{2}\xi^2\zeta^2 + \frac{1}{2}\xi^2\zeta - \frac{1}{2}\eta^2\zeta^2 + \frac{1}{2}\eta^2\zeta + \frac{1}{2}\zeta^2 - \frac{1}{2}\zeta \\
l_{13}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2\zeta + \frac{1}{4}\xi^2\zeta^2 - \frac{1}{4}\xi^2\zeta - \frac{1}{4}\xi\eta^2\zeta^2 + \frac{1}{4}\xi\eta^2\zeta + \frac{1}{4}\xi\zeta^2 - \frac{1}{4}\xi\zeta \\
l_{22}^{(3)} &= \frac{1}{2}\xi^2\eta^2\zeta^2 - \frac{1}{2}\xi^2\eta^2 - \frac{1}{2}\xi^2\zeta^2 + \frac{1}{2}\xi^2 + \frac{1}{2}\xi\eta^2\zeta^2 - \frac{1}{2}\xi\eta^2 - \frac{1}{2}\xi\zeta^2 + \frac{1}{2}\xi \\
l_{27}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 - \frac{1}{4}\xi^2\eta^2\zeta + \frac{1}{4}\xi^2\zeta^2 + \frac{1}{4}\xi^2\zeta - \frac{1}{4}\xi\eta^2\zeta^2 - \frac{1}{4}\xi\eta^2\zeta + \frac{1}{4}\xi\zeta^2 + \frac{1}{4}\xi\zeta \\
l_{26}^{(3)} &= \frac{1}{2}\xi^2\eta^2\zeta^2 + \frac{1}{2}\xi^2\eta^2\zeta - \frac{1}{2}\xi^2\zeta^2 - \frac{1}{2}\xi^2\zeta - \frac{1}{2}\eta^2\zeta^2 - \frac{1}{2}\eta^2\zeta + \frac{1}{2}\zeta^2 + \frac{1}{2}\zeta \\
l_{17}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 - \frac{1}{4}\xi^2\eta^2\zeta + \frac{1}{4}\xi^2\zeta^2 + \frac{1}{4}\xi^2\zeta + \frac{1}{4}\xi\eta^2\zeta^2 + \frac{1}{4}\xi\eta^2\zeta - \frac{1}{4}\xi\zeta^2 - \frac{1}{4}\xi\zeta \\
l_{18}^{(3)} &= \frac{1}{2}\xi^2\eta^2\zeta^2 - \frac{1}{2}\xi^2\eta^2 - \frac{1}{2}\xi^2\zeta^2 + \frac{1}{2}\xi^2 - \frac{1}{2}\xi\eta^2\zeta^2 + \frac{1}{2}\xi\eta^2 + \frac{1}{2}\xi\zeta^2 - \frac{1}{2}\xi \\
l_{14}^{(3)} &= -\xi^2\eta^2\zeta^2 + \xi^2\eta^2 + \xi^2\zeta^2 - \xi^2 + \eta^2\zeta^2 - \eta^2 - \zeta^2 + 1 \\
l_7^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 - \frac{1}{8}\xi^2\eta^2\zeta + \frac{1}{8}\xi^2\eta\zeta^2 - \frac{1}{8}\xi^2\eta\zeta - \frac{1}{8}\xi\eta^2\zeta^2 + \frac{1}{8}\xi\eta^2\zeta - \frac{1}{8}\xi\eta\zeta^2 + \frac{1}{8}\xi\eta\zeta \\
l_6^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2\zeta - \frac{1}{4}\xi^2\eta\zeta^2 + \frac{1}{4}\xi^2\eta\zeta + \frac{1}{4}\eta^2\zeta^2 - \frac{1}{4}\eta^2\zeta + \frac{1}{4}\eta\zeta^2 - \frac{1}{4}\eta\zeta \\
l_5^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 - \frac{1}{8}\xi^2\eta^2\zeta + \frac{1}{8}\xi^2\eta\zeta^2 - \frac{1}{8}\xi^2\eta\zeta + \frac{1}{8}\xi\eta^2\zeta^2 - \frac{1}{8}\xi\eta^2\zeta + \frac{1}{8}\xi\eta\zeta^2 - \frac{1}{8}\xi\eta\zeta \\
l_{14}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2 - \frac{1}{4}\xi^2\eta\zeta^2 + \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2\zeta^2 + \frac{1}{4}\xi\eta^2 - \frac{1}{4}\xi\eta\zeta^2 + \frac{1}{4}\xi\eta
\end{aligned}
$$

$$
\begin{aligned}
l_{23}^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 + \frac{1}{8}\xi^2\eta^2\zeta + \frac{1}{8}\xi^2\eta\zeta^2 + \frac{1}{8}\xi^2\eta\zeta + \frac{1}{8}\xi\eta^2\zeta^2 + \frac{1}{8}\xi\eta^2\zeta + \frac{1}{8}\xi\eta\zeta^2 + \frac{1}{8}\xi\eta\zeta \\
l_{24}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 - \frac{1}{4}\xi^2\eta^2\zeta - \frac{1}{4}\xi^2\eta\zeta^2 - \frac{1}{4}\xi^2\eta\zeta + \frac{1}{4}\eta^2\zeta^2 + \frac{1}{4}\eta^2\zeta + \frac{1}{4}\eta\zeta^2 + \frac{1}{4}\eta\zeta \\
l_{25}^{(3)} &= \frac{1}{8}\xi^2\eta^2\zeta^2 + \frac{1}{8}\xi^2\eta^2\zeta + \frac{1}{8}\xi^2\eta\zeta^2 + \frac{1}{8}\xi^2\eta\zeta - \frac{1}{8}\xi\eta^2\zeta^2 - \frac{1}{8}\xi\eta^2\zeta - \frac{1}{8}\xi\eta\zeta^2 - \frac{1}{8}\xi\eta\zeta \\
l_{16}^{(3)} &= -\frac{1}{4}\xi^2\eta^2\zeta^2 + \frac{1}{4}\xi^2\eta^2 - \frac{1}{4}\xi^2\eta\zeta^2 + \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2\zeta^2 - \frac{1}{4}\xi\eta^2 + \frac{1}{4}\xi\eta\zeta^2 - \frac{1}{4}\xi\eta \\
l_{15}^{(3)} &= \frac{1}{2}\xi^2\eta^2\zeta^2 - \frac{1}{2}\xi^2\eta^2 + \frac{1}{2}\xi^2\eta\zeta^2 - \frac{1}{2}\xi^2\eta - \frac{1}{2}\eta^2\zeta^2 + \frac{1}{2}\eta^2 - \frac{1}{2}\eta\zeta^2 + \frac{1}{2}\eta
\end{aligned}
$$





Conclusiones

Se implementó la forma de cálculo en Python [3], usando IPython notebook [4], Numpy [5], Sympy [6], Matplotlib [7] y Mayavi [8]; también se realizo una tabla comparativa de los tiempos de ejecución de las funciones:

Elemento 1D		Elemento 2D		Elemento 3D	
2 nodos	0.003 seg.	4 nodos	0.021 seg.	8 nodos	0.060 seg.
3 nodos	0.003 seg.	9 nodos	0.006 seg.	27 nodos	0.014 seg.
4 nodos	0.055 seg.	16 nodos	0.345 seg.	64 nodos	4.637 seg.
5 nodos	0.074 seg.	25 nodos	0.469 seg.	125 nodos	7.197 seg.
6 nodos	0.167 seg.	36 nodos	1.530 seg.	216 nodos	57.337 seg.

Para generar elementos de mayor grado y reducir el tiempo de cálculo se optimizará el código y se usara Numba [9].
Se estudiarán las propiedades de las matrices $l^{(2)}$:

- Rango
- Traza
- Determinante

En el caso de las matrices $l^{(3)}$:

- Hiperdeterminante

Los archivos del trabajo se encuentran en `https://github.com/ClaudioVZ/Teoria-FEM-Python`.

Referencias

[1] O.C.Zienkiewicz; R.L.Taylor. *El Método de los Elementos Finitos Volumen 1.* McGrall-Hill; CIMNE. 1994.

[2] ClaudioVZ. *Hipermatriz.* 2013.

[3] Lenguaje de programación `http://www.python.org/`

[4] Intérprete interactivo `http://ipython.org/`

[5] Biblioteca de funciones matemáticas y operaciones con arreglos `http://www.numpy.org/`

[6] Biblioteca para matemática simbólica `http://sympy.org/en/index.html`

[7] Biblioteca para gráficos 2D y 3D `http://matplotlib.org/`

[8] Biblioteca para gráficos 3D `http://code.enthought.com/projects/mayavi/`

[9] Compilador jit especializado `http://numba.pydata.org/`