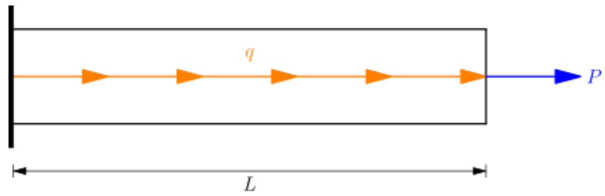
Ejemplo 3



Resolver

$$EA\,\frac{d^2u}{dx^2} + q = 0$$
$$u(0) = 0$$
$$u'(L) = \frac{P}{EA}$$

Solución exacta

$$u(x) = \frac{qL + P}{EA}x - \frac{q}{2EA}x^2$$

Solución aproximada cuadrática

La forma débil de la ecuación diferencial es

$$\int_0^L R(x)\,W(x)\,dx = \int_0^L \left(EA\,\frac{d^2\hat{u}}{dx^2} + q\right)W\,dx = 0$$

reduciendo el grado de las derivadas

$$\int_0^L \frac{dW}{dx}\,EA\,\frac{d\hat{u}}{dx}\,dx = \int_0^L W\,q\,dx + W(L)\,EA\frac{d\hat{u}(L)}{dx} - W(0)\,EA\frac{d\hat{u}(0)}{dx}$$

usando bases cuadráticas en coordenadas locales

$$u(x) \approx \hat{u}(x) = u_1\left(1 - \frac{3}{L}x + \frac{2}{L^2}x^2\right) + u_2\left(\frac{4}{L}x - \frac{4}{L^2}x^2\right) + u_3\left(-\frac{1}{L}x + \frac{2}{L^2}x^2\right)$$

$\hat{u}_x$ es

$$\frac{d\hat{u}}{dx} = -\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x$$

las funciones ponderadas son

$$W_1 = \frac{d\hat{u}}{du_1} = 1 - \frac{3}{L}x + \frac{2}{L^2}x^2$$
$$W_2 = \frac{d\hat{u}}{du_2} = \frac{4}{L}x - \frac{4}{L^2}x^2$$
$$W_3 = \frac{d\hat{u}}{du_3} = -\frac{1}{L}x + \frac{2}{L^2}x^2$$

formando el sistema de ecuaciones

$$\int_0^L \frac{dW_1}{dx}\,EA\,\frac{d\hat{u}}{dx}\,dx = \int_0^L W_1\,q\,dx + W_1(L)\,EA\frac{d\hat{u}(L)}{dx} - W_1(0)\,EA\frac{d\hat{u}(0)}{dx}$$
$$\int_0^L \frac{dW_2}{dx}\,EA\,\frac{d\hat{u}}{dx}\,dx = \int_0^L W_2\,q\,dx + W_2(L)\,EA\frac{d\hat{u}(L)}{dx} - W_2(0)\,EA\frac{d\hat{u}(0)}{dx}$$
$$\int_0^L \frac{dW_3}{dx}\,EA\,\frac{d\hat{u}}{dx}\,dx = \int_0^L W_3\,q\,dx + W_3(L)\,EA\frac{d\hat{u}(L)}{dx} - W_3(0)\,EA\frac{d\hat{u}(0)}{dx}$$

funciones ponderadas y sus derivadas

$$W_1 = 1 - \frac{3}{L}x + \frac{2}{L^2}x^2 \quad \frac{dW_1}{dx} = -\frac{3}{L} + \frac{4}{L^2}x \qquad W_2 = \frac{4}{L}x - \frac{4}{L^2}x^2 \quad \frac{dW_2}{dx} = \frac{4}{L} - \frac{8}{L^2}x$$
$$W_3 = -\frac{1}{L}x + \frac{2}{L^2}x^2 \qquad \frac{dW_3}{dx} = -\frac{1}{L} + \frac{4}{L^2}x$$

valores de las funciones ponderadas en los nodos

$$W_1(L) = 0 \quad W_1(0) = 1 \qquad W_2(L) = 0 \quad W_2(0) = 0$$
$$W_3(L) = 1 \quad W_3(0) = 0$$

fuerzas en los nodos

$$EA\frac{d\hat{u}(L)}{dx} = F_3 \qquad EA\frac{d\hat{u}(0)}{dx} = F_1$$

reemplazando

$$\int_0^L \left(\frac{3}{L} + \frac{4}{L^2}x\right) EA\left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x\right) dx = \int_0^L \left(1 - \frac{3}{L}x + \frac{2}{L^2}x^2\right) q\,dx + 0(F_3) - 1(F_1)$$
$$\int_0^L \left(\frac{4}{L} - \frac{8}{L^2}x\right) EA\left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x\right) dx = \int_0^L \left(\frac{4}{L}x - \frac{4}{L^2}x^2\right) q\,dx + 0(F_3) - 0(F_1)$$
$$\int_0^L \left(-\frac{1}{L} + \frac{4}{L^2}x\right) EA\left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x\right) dx = \int_0^L \left(-\frac{1}{L}x + \frac{2}{L^2}x^2\right) q\,dx + 1(F_3) - 0(F_1)$$

reordenando

$$\frac{EA}{L}\int_0^L \left(3 + \frac{4}{L}x\right)\left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x\right) dx = \int_0^L \left(1 - \frac{3}{L}x + \frac{2}{L^2}x^2\right) q\,dx - F_1$$
$$\frac{EA}{L}\int_0^L \left(4 - \frac{8}{L}x\right)\left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x\right) dx = \int_0^L \left(\frac{4}{L}x - \frac{4}{L^2}x^2\right) q\,dx$$
$$\frac{EA}{L}\int_0^L \left(-1 + \frac{4}{L}x\right)\left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x\right) dx = \int_0^L \left(-\frac{1}{L}x + \frac{2}{L^2}x^2\right) q\,dx + F_3$$

integrando

$$\frac{EA}{L}\left(\frac{7}{3}u_1 - \frac{8}{3}u_2 + \frac{1}{3}u_3\right) = \frac{qL}{6} - F_1$$
$$\frac{EA}{L}\left(-\frac{8}{3}u_1 + \frac{16}{3}u_2 - \frac{8}{3}u_3\right) = \frac{2qL}{3}$$
$$\frac{EA}{L}\left(\frac{1}{3}u_1 - \frac{8}{3}u_2 + \frac{7}{3}u_3\right) = \frac{qL}{6} + F_3$$

en forma matricial

$$\frac{EA}{L}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{qL}{6} \\ \frac{2qL}{3} \\ \frac{qL}{6} \end{bmatrix} + \begin{bmatrix} -F_1 \\ 0 \\ F_3 \end{bmatrix}$$

reemplazando fuerzas y desplazamientos

$$\frac{EA}{L}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{qL}{6} \\ \frac{2qL}{3} \\ \frac{qL}{6} \end{bmatrix} + \begin{bmatrix} -F_1 \\ 0 \\ P \end{bmatrix}$$

resolviendo

$$u_2 = \frac{3qL^2 + 4PL}{8EA}$$
$$u_3 = \frac{qL^2 + 2PL}{2EA}$$
$$F_1 = qL + P$$

reemplazando en la solución aproximada

$$\hat{u}(x) = \left(\frac{3qL^2 + 4PL}{8EA}\right)\left(\frac{4}{L}x - \frac{4}{L^2}x^2\right) + \left(\frac{qL^2 + 2PL}{2EA}\right)\left(-\frac{1}{L}x + \frac{2}{L^2}x^2\right) = \frac{qL + P}{EA}x - \frac{q}{2EA}x^2$$