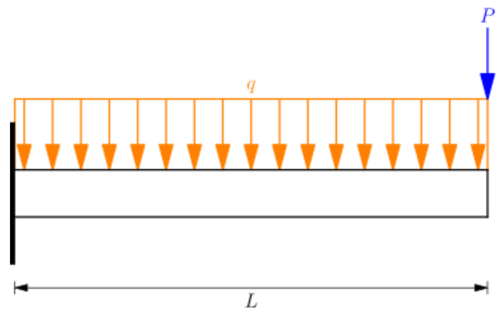
Ejemplo 5



Resolver

$$EI\,\frac{d^4v}{dx^4} + q = 0$$
$$v(0) = 0 \qquad EIv''(L) = 0$$
$$v'(0) = 0 \qquad EIv'''(L) = P$$

Solución exacta

$$v(x) = -\frac{qL^2 + 2PL}{4EI}x^2 + \frac{qL+P}{6EI}x^3 - \frac{q}{24EI}x^4$$

Solución aproximada cúbica

La forma débil de la ecuación diferencial es

$$\int_0^L R(x)\,W(x)\,dx = \int_0^L \left(EI\,\frac{d^4\hat{v}}{dx^4} + q\right)W\,dx = 0$$

reduciendo el grado de las derivadas

$$\int_0^L \frac{d^2W}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W\,q\,dx - W(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$

usando bases cúbicas en coordenadas locales

$$v(x) \approx \hat{v}(x) = v_1\left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3\right) + \theta_1\left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3\right) + v_2\left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) + \theta_2\left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right)$$

$\hat{v}_{xx}$ es

$$\frac{d^2\hat{v}}{dx^2} = \left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2$$

las funciones ponderadas son

$$W_1 = \frac{d\hat{v}}{dv_1} = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3$$
$$W_2 = \frac{d\hat{v}}{d\theta_1} = x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3$$
$$W_3 = \frac{d\hat{v}}{dv_2} = \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3$$
$$W_4 = \frac{d\hat{v}}{d\theta_2} = -\frac{1}{L}x^2 + \frac{1}{L^2}x^3$$

formando el sistema de ecuaciones

$$\int_0^L \frac{d^2W_1}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_1\,q\,dx - W_1(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_1(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_1(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_1(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$
$$\int_0^L \frac{d^2W_2}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_2\,q\,dx - W_2(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_2(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_2(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_2(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$
$$\int_0^L \frac{d^2W_3}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_3\,q\,dx - W_3(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_3(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_3(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_3(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$
$$\int_0^L \frac{d^2W_4}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_4\,q\,dx - W_4(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_4(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_4(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_4(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$

funciones ponderadas y sus derivadas

$$W_1 = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \quad \frac{d^2W_1}{dx^2} = -\frac{6}{L^2} + \frac{12}{L^3}x \qquad W_2 = x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 \quad \frac{d^2W_2}{dx^2} = -\frac{4}{L} + \frac{6}{L^2}x$$
$$W_3 = \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \qquad \frac{d^2W_3}{dx^2} = \frac{6}{L^2} - \frac{12}{L^3}x \qquad W_4 = -\frac{1}{L}x^2 + \frac{1}{L^2}x^3 \qquad \frac{d^2W_4}{dx^2} = -\frac{2}{L} + \frac{6}{L^2}x$$

valores de las funciones ponderadas en los nodos

$$W_1(L) = 0 \quad W_1(0) = 1 \qquad \frac{dW_1(L)}{dx} = 0 \quad \frac{dW_1(0)}{dx} = 0$$
$$W_2(L) = 0 \quad W_2(0) = 0 \qquad \frac{dW_2(L)}{dx} = 0 \quad \frac{dW_2(0)}{dx} = 1$$
$$W_3(L) = 1 \quad W_3(0) = 0 \qquad \frac{dW_3(L)}{dx} = 0 \quad \frac{dW_3(0)}{dx} = 0$$
$$W_4(L) = 0 \quad W_4(0) = 0 \qquad \frac{dW_4(L)}{dx} = 1 \quad \frac{dW_4(0)}{dx} = 0$$

cortante y momento en los nodos

$$EI\frac{d^3\hat{v}(L)}{dx^3} = V_2 \qquad EI\frac{d^3\hat{v}(0)}{dx^3} = V_1$$
$$EI\frac{d^2\hat{v}(L)}{dx^2} = M_2 \qquad EI\frac{d^2\hat{v}(0)}{dx^2} = M_1$$

reemplazando

$$\int_0^L \left(-\frac{6}{L^2} + \frac{12}{L^3}\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3\right) q\,dx - 0(V_2) + 1(V_1) + 0(M_2) - 0(M_1)$$
$$\int_0^L \left(-\frac{4}{L} + \frac{6}{L^2}x\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3\right) q\,dx - 0(V_2) + 0(V_1) + 0(M_2) - 1(M_1)$$
$$\int_0^L \left(\frac{6}{L^2} - \frac{12}{L^3}x\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) q\,dx - 1(V_2) + 0(V_1) + 0(M_2) - 0(M_1)$$
$$\int_0^L \left(-\frac{2}{L} + \frac{6}{L^2}x\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right) q\,dx - 0(V_2) + 0(V_1) + 1(M_2) - 0(M_1)$$

integrando

$$\begin{aligned}
\frac{EI}{L^3}(12v_1+6L\theta_1-12v_2+6L\theta_2) &= -\frac{qL}{2}+V_1\\
\frac{EI}{L^3}(6Lv_1+4L^2\theta_1-6Lv_2+2L^2\theta_2) &= -\frac{qL^2}{12}-M_1\\
\frac{EI}{L^3}(-12v_1-6L\theta_1+12v_2-6L\theta_2) &= -\frac{qL}{2}-V_2\\
\frac{EI}{L^3}(6Lv_1+2L^2\theta_1-6Lv_2+4L^2\theta_2) &= \frac{qL^2}{12}+M_2
\end{aligned}$$

en forma matricial

$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}-\frac{qL}{2}\\ -\frac{qL^2}{12}\\ -\frac{qL}{2}\\ \frac{qL^2}{12}\end{bmatrix}+\begin{bmatrix}V_1\\ -M_1\\ -V_2\\ M_2\end{bmatrix}$$

reemplazando fuerzas y desplazamientos

$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}0\\ 0\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}-\frac{qL}{2}\\ -\frac{qL^2}{12}\\ -\frac{qL}{2}\\ \frac{qL^2}{12}\end{bmatrix}+\begin{bmatrix}V_1\\ -M_1\\ -P\\ 0\end{bmatrix}$$

resolviendo

$$\begin{aligned}
v_2 &= -\frac{3qL^4+8PL^3}{24EI}\\
\theta_2 &= -\frac{qL^3+3PL^2}{6EI}\\
V_1 &= qL+P\\
M_1 &= -\frac{qL^2+2PL}{2}
\end{aligned}$$

reemplazando en la solución aproximada

$$\hat{v}(x)=\left(-\frac{3qL^4+8PL^3}{24EI}\right)\left(\frac{3}{L^2}x^2-\frac{2}{L^3}x^3\right)+\left(-\frac{qL^3+3PL^2}{6EI}\right)\left(-\frac{1}{L}x^2+\frac{1}{L^2}x^3\right)=-\frac{5qL^2+12PL}{24EI}x^2+\frac{qL+2P}{12EI}x^3$$