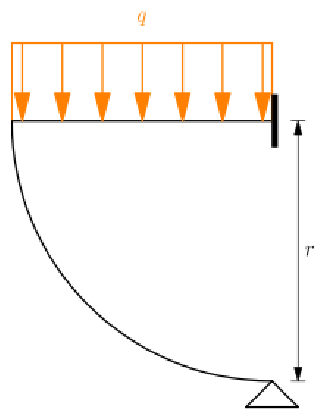
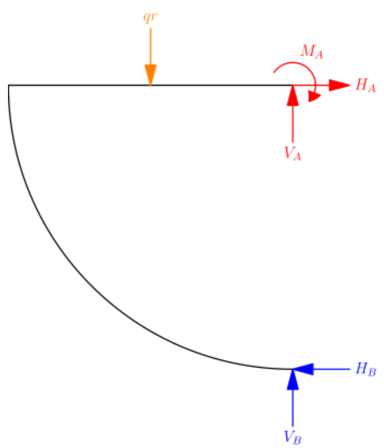
Introducción a elementos finitos
Primer Parcial I-2016

1. Resolver la estructura con E, I, A constantes por el método de Castigliano



Solución

Estructura equivalente

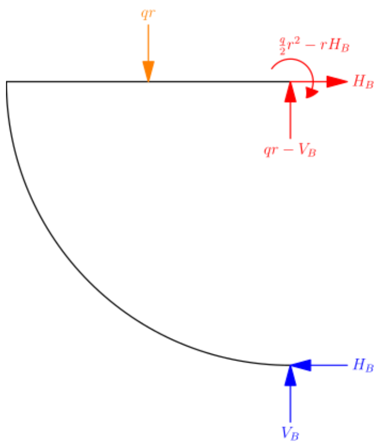


De la anterior estructura se obtienen tres ecuaciones con cinco incógnitas.

$$\begin{aligned} H_A - H_B &= 0 \\ -qr + V_A + V_B &= 0 \\ \frac{q}{2}r^2 - H_B r - M_A &= 0 \end{aligned}$$

Se parametrizarán H_A, V_A y M_A.

$$\begin{aligned} H_A &= H_B \\ V_A &= qr - V_B \\ M_A &= \frac{q}{2}r^2 - rH_B \end{aligned}$$



Esfuerzos internos de la viga

$$\begin{aligned} N &= H_B \\ V &= qx - (qr - V_B) = qx - qr + V_B \\ M &= -\frac{q}{2}x^2 + (qr - V_B)x - \left(\frac{q}{2}r^2 - H_B r\right) \\ &= -\frac{q}{2}x^2 + (qr - V_B)x - \frac{q}{2}r^2 + H_B r \end{aligned}$$

Esfuerzos internos del arco

$$
\begin{aligned}
N &= -H_B \cos\theta - V_B \sin\theta \\
V &= H_B \sin\theta - V_B \cos\theta \\
M &= -H_B r \sin\theta - V_B (r - r\cos\theta) = -H_B r \sin\theta + V_B r \cos\theta - V_B r
\end{aligned}
$$

Para simplificar el cálculo solo usaré la energía de deformación por flexión

$$
U_i = \int_0^r \frac{M^2}{2EI}\,dx + \int_0^s \frac{M^2}{2EI}\,ds = \int_0^r \frac{M^2}{2EI}\,dx + \int_0^{\frac{\pi}{2}} \frac{M^2}{2EI}\,r\,d\theta
$$

Reemplazando

$$
\begin{aligned}
U_i = &\frac{1}{2EI}\int_0^r \left[\frac{q}{2}x^2 + (qr - V_B)x - \frac{q}{2}r^2 + H_B r\right]^2 dx \\
&+ \frac{1}{2EI}\int_0^{\frac{\pi}{2}} (-H_B r \sin\theta + V_B r\cos\theta - V_B r)^2\, r\,d\theta
\end{aligned}
$$

Integrando

$$
\begin{aligned}
U_i = \frac{r^3}{40EI}\Big[q^2 r^2 - \frac{20\,qr}{3}\Big(H_B - \frac{1}{4}V_B\Big) + \Big(15\pi - \frac{100}{3}\Big)V_B^2 - 40 H_B V_B \\
+ 5(\pi + 4)H_B^2\Big]
\end{aligned}
$$

Minimizando

$$
\begin{aligned}
\frac{\partial U_i}{\partial H_B} &= \frac{r^3}{6EI}\Big[\Big(\frac{3\pi}{2} + 6\Big)H_B - 6V_B - qr\Big] = 0 \\
\frac{\partial U_i}{\partial V_B} &= \frac{r^3}{24EI}[-24H_B + (18\pi - 40)V_B + qr] = 0
\end{aligned}
$$

Formando el sistema de ecuaciones

$$
\begin{aligned}
(3\pi + 12)H_B - 12V_B &= 2qr \\
-24H_B + (18\pi - 40)V_B &= -qr
\end{aligned}
$$

Resolviendo

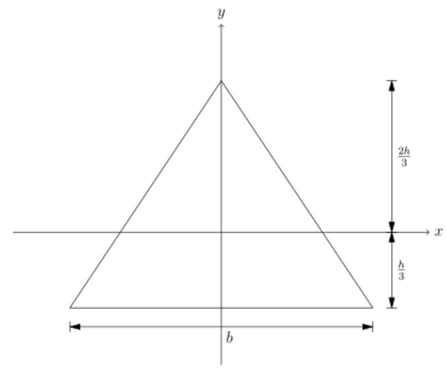
$$
\begin{aligned}
H_B &= \frac{2qr(9\pi - 23)}{27\pi^2 + 48\pi - 384} \\
V_B &= \frac{qr(12 - \pi)}{18\pi^2 + 32\pi - 256}
\end{aligned}
$$

Reemplazando en las demás reacciones

$$
\begin{aligned}
H_A &= H_B = \frac{2qr(9\pi - 23)}{27\pi^2 + 48\pi - 384} \\
V_A &= qr - V_B = qr - \frac{qr(12 - \pi)}{18\pi^2 + 32\pi - 256} = \frac{qr(18\pi^2 + 33\pi - 268)}{18\pi^2 + 32\pi - 256} \\
M_A &= \frac{q}{2}r^2 - rH_B = \frac{q}{2}r^2 - r\Big[\frac{2qr(9\pi - 23)}{27\pi^2 + 48\pi - 384}\Big] = \frac{qr^2(27\pi^2 + 12\pi - 292)}{54\pi^2 + 96\pi - 768}
\end{aligned}
$$

2. Calcular el factor de forma de una sección triangular

Solución



El momento estático es

$$
Q = \int_A y\,dA = \int_0^{\frac{2}{3}b - \frac{b}{h}y} \int_y^{\frac{2}{3}h - \frac{h}{b}x} y\,dy\,dx
$$

Integrando respecto de y

$$
Q = \int_0^{\frac{2}{3}b - \frac{b}{h}y} \frac{y^2}{2}\Big|_y^{\frac{2}{3}h - \frac{h}{b}x} dx = \int_0^{\frac{2}{3}b - \frac{b}{h}y} \frac{2h^2}{9} - \frac{y^2}{2} - \frac{2h^2}{3b}x + \frac{h^2}{2b^2}x^2\ dx
$$

Integrando respecto de x

$$Q = \left[\left(\frac{2h^2}{9} - \frac{y^2}{2}\right)x - \frac{h^2}{3b}x^2 + \frac{h^2}{6b^2}x^3\right]\Bigg|_0^{\frac{2}{3}b-\frac{b}{h}y} = \frac{4bh^2}{81} - \frac{b}{3}y^2 + \frac{b}{3h}y^3$$

Factor de forma

$$k = \frac{A}{I^2}\int_A \frac{Q^2}{b^2}\,dA = \frac{A}{I^2}\int_{-\frac{h}{3}}^{\frac{2h}{3}} \frac{Q^2}{x^2}\,x\,dy = \frac{A}{I^2}\int_{-\frac{h}{3}}^{\frac{2h}{3}} \frac{Q^2}{x}\,dy$$

Reemplazando valores

$$k = \frac{\frac{bh}{2}}{\left(\frac{bh^3}{36}\right)^2}\int_{-\frac{h}{3}}^{\frac{2h}{3}} \frac{\left(\frac{4bh^2}{81} - \frac{b}{3}y^2 + \frac{b}{3h}y^3\right)^2}{\frac{2}{3}b - \frac{b}{h}y}\,dy$$

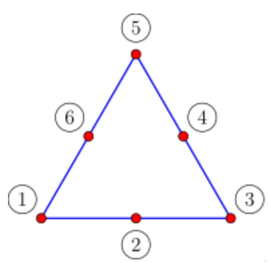
Integrando

$$k = \frac{648}{bh^5}\left(\frac{8bh^4}{2187}y + \frac{2bh^3}{729}y^2 - \frac{10bh^2}{729}y^3 - \frac{bh}{324}y^4 + \frac{4b}{135}y^5 - \frac{b}{54h}y^6\right)\Bigg|_{-\frac{h}{3}}^{\frac{2h}{3}}$$

Simplificando

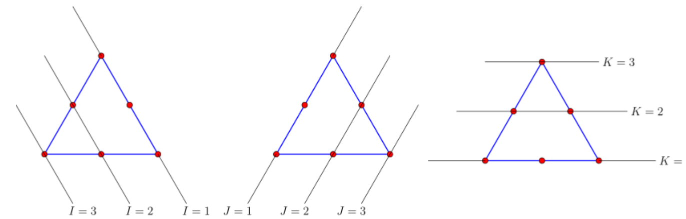
$$k = \frac{648}{bh^5}\left(\frac{bh^5}{540}\right) = \frac{6}{5}$$

3. Calcular las funciones de forma $\boldsymbol{N}$



Solución

Numeración de nodos



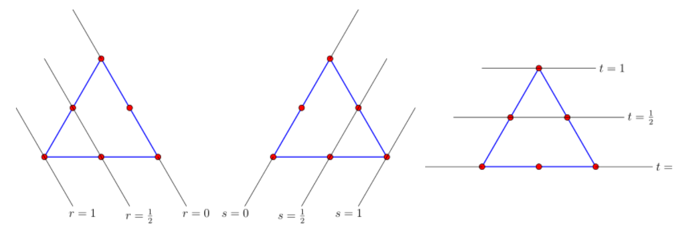
$$ⓘ$$

$①=[I_1,J_1,K_1]=[3,1,1]$ $④=[I_4,J_4,K_4]=[1,2,2]$

$②=[I_2,J_2,K_2]=[2,2,1]$ $⑤=[I_5,J_5,K_5]=[1,1,3]$

$③=[I_3,J_3,K_3]=[1,3,1]$ $⑥=[I_6,J_6,K_6]=[2,1,2]$

Coordenadas de nodos



① $= [r_3, s_1, t_1] = [1, 0, 0]$ ④ $= [r_1, s_2, t_2] = \left[0, \frac{1}{2}, \frac{1}{2}\right]$

② $= [r_2, s_2, t_1] = \left[\frac{1}{2}, \frac{1}{2}, 0\right]$ ⑤ $= [r_1, s_1, t_3] = [0, 0, 1]$

③ $= [r_1, s_3, t_1] = [0, 1, 0]$ ⑥ $= [r_2, s_1, s_2] = \left[\frac{1}{2}, 0, \frac{1}{2}\right]$

Nodo ①

Reemplazando numeración y coordenadas

$$T_3(r) = \frac{r - r_2}{r_3 - r_2}\cdot\frac{r - r_1}{r_3 - r_1} = \frac{r - \frac{1}{2}}{1 - \frac{1}{2}}\cdot\frac{r - 0}{1 - 0} = r(2r - 1)$$

$T_1(s) = 1$

$T_1(t) = 1$

Reemplazando polinomios

$$N_1 = T_3 T_1 T_1 = r(2r-1)\cdot 1\cdot 1 = r(2r-1)$$

Nodo ②

Reemplazando numeración y coordenadas

$$T_2(r) = \frac{r-r_1}{r_2-r_1} = \frac{r-0}{\frac{1}{2}-0} = 2r$$
$$T_2(s) = \frac{s-s_1}{s_2-s_1} = \frac{s-0}{\frac{1}{2}-0} = 2s$$
$$T_1(t) = 1$$

Reemplazando polinomios

$$N_2 = T_2 T_2 T_1 = 2r\cdot 2s\cdot 1 = 4rs$$

Nodo ③

Reemplazando numeración y coordenadas

$$T_1(r) = 1$$
$$T_3(s) = \frac{s-s_2}{s_3-s_2}\cdot\frac{s-s_1}{s_3-s_1} = \frac{s-\frac{1}{2}}{1-\frac{1}{2}}\cdot\frac{s-0}{1-0} = s(2s-1)$$
$$T_1(t) = 1$$

Reemplazando polinomios

$$N_3 = T_1 T_3 T_1 = 1\cdot s(2s-1)\cdot 1 = s(2s-1)$$

Nodo ④

Reemplazando numeración y coordenadas

$$T_1(r) = 1$$
$$T_2(s) = \frac{s-s_1}{s_2-s_1} = \frac{s-0}{\frac{1}{2}-0} = 2s$$
$$T_2(t) = \frac{t-t_1}{t_2-t_1} = \frac{t-0}{\frac{1}{2}-0} = 2t$$

Reemplazando polinomios

$$N_4 = T_1 T_2 T_2 = 1\cdot 2s\cdot 2t = 4st$$

Nodo ⑤

Reemplazando numeración y coordenadas

$$T_1(r) = 1$$
$$T_1(s) = 1$$
$$T_3(t) = \frac{t-t_2}{t_3-t_2}\cdot\frac{t-t_1}{t_3-t_1} = \frac{t-\frac{1}{2}}{1-\frac{1}{2}}\cdot\frac{t-0}{1-0} = t(2t-1)$$

Reemplazando polinomios

$$N_5 = T_1 T_1 T_3 = 1\cdot 1\cdot t(2t-1) = t(2t-1)$$

Nodo ⑥

Reemplazando numeración y coordenadas

$$T_2(r) = \frac{r-r_1}{r_2-r_1} = \frac{r-0}{\frac{1}{2}-0} = 2r$$
$$T_1(s) = 1$$
$$T_2(t) = \frac{t-t_1}{t_2-t_1} = \frac{t-0}{\frac{1}{2}-0} = 2t$$

Reemplazando polinomios

$$N_6 = T_2 T_1 T_2 = 2r\cdot 1\cdot 2t = 4rt$$