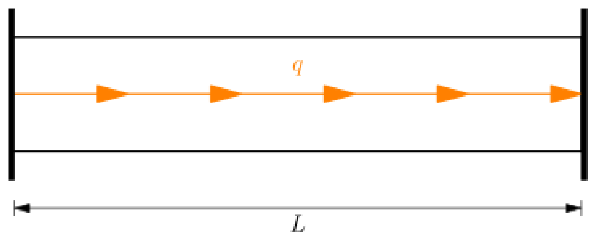
Introducción a elementos finitos
Primer Parcial I-2017

1. Resolver la estructura con E, A constantes por el método de Rayleigh-Ritz



Solución

La solución exacta es un polinomio de segundo grado, la aproximación del campo de desplazamientos será

$$u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

Reemplazando $u(0) = 0$ y $u(L) = 0$

$$\alpha_0 + \alpha_1(0) + \alpha_2(0)^2 = 0$$
$$\alpha_0 + \alpha_1(L) + \alpha_2(L)^2 = 0$$

Resolviendo

$$\alpha_0 = 0$$
$$\alpha_1 = -\alpha_2 L$$

Reemplazando en el campo de desplazamientos

$$u = -\alpha_2 L x + \alpha_2 x^2$$

Carga distribuida

$$F = q$$

La deformación unitaria es

$$\varepsilon = \frac{du}{dx} = -\alpha_2 L + 2\alpha_2 x$$

El funcional de energía es

$$\pi = \int_0^L \frac{1}{2}\,\varepsilon\,\sigma\,dV - \int_0^L F\,u\,dx = \int_0^L \frac{1}{2} EA\,\varepsilon^2\,dx - \int_0^L F\,u\,dx$$

Reemplazando

$$\pi = \int_0^L \frac{EA}{2}\left(-\alpha_2 L + 2\alpha_2 x\right)^2 dx - \int_0^L q\left(-\alpha_2 L x + \alpha_2 x^2\right) dx$$

Integrando

$$\pi = \frac{EAL^3}{6}\,\alpha_2^2 + \frac{qL^3}{6}\,\alpha_2$$

Minimizando el funcional

$$\frac{\partial \pi}{\partial \alpha_2} = \frac{EAL^3}{3}\,\alpha_2 + \frac{qL^3}{6} = 0$$

Reordenando

$$\frac{EAL^3}{3}\,\alpha_2 = -\frac{qL^3}{6}$$

Resolviendo

$$\alpha_2 = -\frac{q}{2EA}$$

Reemplazando en u

$$u = \frac{qL}{2EA}\,x - \frac{q}{2EA}\,x^2 = \frac{q}{2EA}\left(Lx - x^2\right)$$

2. Calcular la integral mediante la cuadratura de Newton-Cotes para $n = 2$

$$I = \int_0^2 x^3 e^x\,dx$$

Solución

$$k = 2 - 1 = 1$$

Calculando r_i

$$\int_{-1}^{+1} P(r)\ r^0\ dr = 0$$
$$\int_{-1}^{+1} P(r)\ r^1\ dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2)\ dr = 0$$
$$\int_{-1}^{+1} (r - r_1)(r - r_2) r\ dr = 0$$

Integrando

$$\left(\frac{1}{3} r^3 - \frac{r_1 + r_2}{2} r^2 + r_1 r_2 r\right)\Bigg|_{-1}^{+1} = 2\left(r_1 r_2 + \frac{1}{3}\right)$$
$$\left(\frac{1}{4} r^4 - \frac{r_1 + r_2}{3} r^3 + \frac{r_1 r_2}{2} r^2\right)\Bigg|_{-1}^{+1} = -\frac{2}{3}(r_1 + r_2)$$

Formando el sistema de ecuaciones

$$r_1 r_2 = -\frac{1}{3}$$
$$r_1 + r_2 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{1}{3}}$$
$$r_2 = \sqrt{\frac{1}{3}}$$

Calculando w_i

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2}\ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1}\ dr$$

Reemplazando

$$w_1 = \int_{-1}^{+1} \frac{r - \sqrt{\frac{1}{3}}}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}}\ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}}\ dr$$

Integrando

$$w_1 = \left(-\frac{\sqrt{3}}{4} r^2 + \frac{1}{2} r\right)\Bigg|_{-1}^{+1} = 1$$
$$w_2 = \left(\frac{\sqrt{3}}{4} r^2 + \frac{1}{2} r\right)\Bigg|_{-1}^{+1} = 1$$

Usando la fórmula

$$I = w_1' f(r_1') + w_2' f(r_2')$$

Puntos de muestreo

$$r_1' = \frac{b + a}{2} + \frac{b - a}{2} r_1 = \frac{2 + 0}{2} + \frac{2 - 0}{2}\left(-\sqrt{\frac{1}{3}}\right) = 0.42265$$
$$r_2' = \frac{b + a}{2} + \frac{b - a}{2} r_2 = \frac{2 + 0}{2} + \frac{2 - 0}{2}\left(\sqrt{\frac{1}{3}}\right) = 1.57735$$

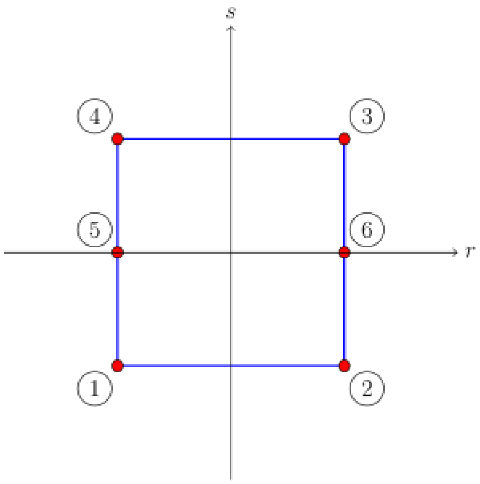
Pesos

$$w_1' = \frac{b - a}{2} w_1 = \frac{2 - 0}{2}(1) = 1$$
$$w_2' = \frac{b - a}{2} w_2 = \frac{2 - 0}{2}(1) = 1$$

Reemplazando

$$I = 1\left(0.42265^3 e^{0.42265}\right) + 1\left(1.57735^3 e^{1.57735}\right) = 19.11806$$

3. Hallar las funciones de interpolación de Lagrange en coordenada naturales



Solución

Coordenadas de los nodos

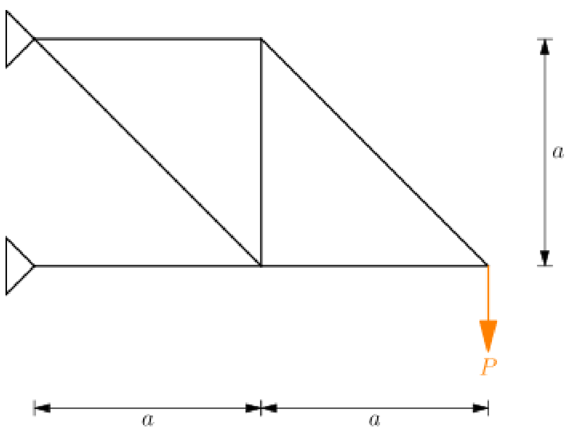
$$\begin{aligned}
1 &= [r_1, s_1] = [-1, -1] & 4 &= [r_4, s_4] = [-1, 1] \\
2 &= [r_2, s_2] = [1, -1] & 5 &= [r_5, s_5] = [-1, 0] \\
3 &= [r_3, s_3] = [1, 1] & 6 &= [r_6, s_6] = [1, 0]
\end{aligned}$$

Reemplazando valores

$$\begin{aligned}
N_1 &= \frac{r - r_2}{r_1 - r_2} \cdot \frac{s - s_5}{s_1 - s_5} \cdot \frac{s - s_4}{s_1 - s_4} \\
&= \frac{r - 1}{-1 - 1} \cdot \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1} = -\frac{1}{4}(r - 1)s(s - 1) \\
N_2 &= \frac{r - r_1}{r_2 - r_1} \cdot \frac{s - s_6}{s_2 - s_6} \cdot \frac{s - s_3}{s_2 - s_3} \\
&= \frac{r - (-1)}{1 - (-1)} \cdot \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1} = \frac{1}{4}(r + 1)s(s - 1)
\end{aligned}$$

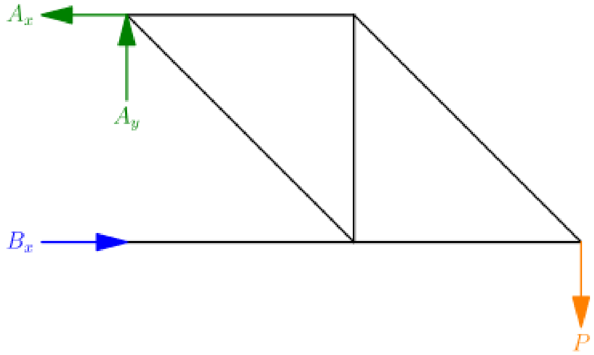
$$\begin{aligned}
N_3 &= \frac{r - r_4}{r_3 - r_4} \cdot \frac{s - s_6}{s_3 - s_6} \cdot \frac{s - s_2}{s_3 - s_2} \\
&= \frac{r - (-1)}{1 - (-1)} \cdot \frac{s - 0}{1 - 0} \cdot \frac{s - (-1)}{1 - (-1)} = \frac{1}{4}(r + 1)s(s + 1) \\
N_4 &= \frac{r - r_3}{r_4 - r_3} \cdot \frac{s - s_5}{s_4 - s_5} \cdot \frac{s - s_1}{s_4 - s_1} \\
&= \frac{r - 1}{-1 - 1} \cdot \frac{s - 0}{1 - 0} \cdot \frac{s - (-1)}{1 - (-1)} = -\frac{1}{4}(r - 1)s(s + 1) \\
N_5 &= \frac{r - r_6}{r_5 - r_6} \cdot \frac{s - s_4}{s_5 - s_4} \cdot \frac{s - s_1}{s_5 - s_1} \\
&= \frac{r - 1}{-1 - 1} \cdot \frac{s - 1}{0 - 1} \cdot \frac{s - (-1)}{0 - (-1)} = \frac{1}{2}(r - 1)(s - 1)(s + 1) \\
N_6 &= \frac{r - r_5}{r_6 - r_5} \cdot \frac{s - s_4}{s_6 - s_4} \cdot \frac{s - s_1}{s_6 - s_1} \\
&= \frac{r - (-1)}{1 - (-1)} \cdot \frac{s - 1}{0 - 1} \cdot \frac{s - (-1)}{0 - (-1)} = -\frac{1}{2}(r + 1)(s - 1)(s + 1)
\end{aligned}$$

4. Resolver la estructura cuyas barras tienen una rigidez EA constante por el método de Castigliano



Solución

Estructura equivalente



Grado de hiperestaticidad externa

$$GHE = NR - NEE = 3 - 3 = 0$$

Grado de hiperestaticidad interna

$$GHI = NE - 2NN + NEE = 6 - 2(5) + 3 = -1$$

Grado de hiperestaticidad total

$$GHT = GHE + GHI = 0 - 1 = -1$$

La estructura es inestable.