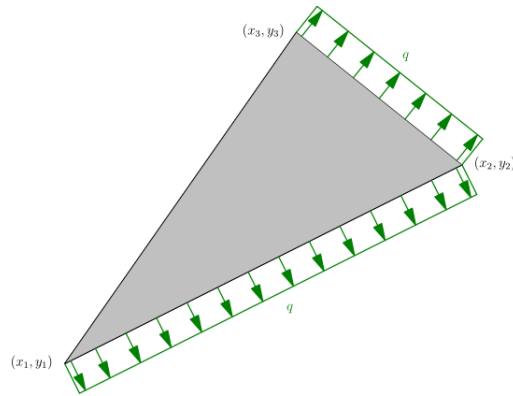


Introducción a elementos finitos

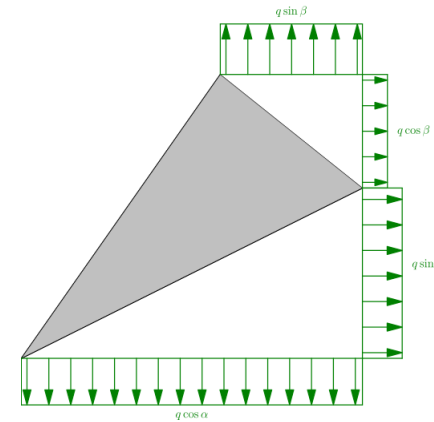
Primer Parcial II-2016

1. Calcular las fuerzas nodales de la placa de espesor t sometida a esfuerzo plano

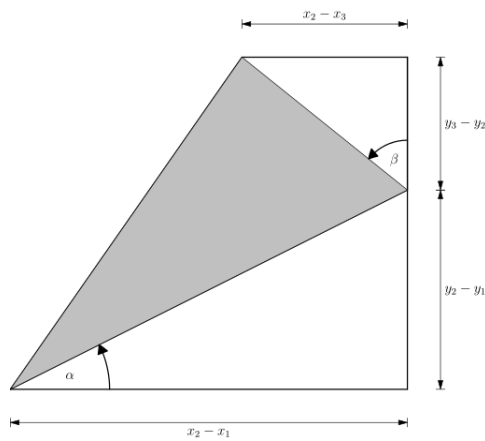


$$\sin \alpha = \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \quad \cos \alpha = \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$$\sin \beta = \frac{x_2 - x_3}{\sqrt{(x_2 - x_3)^2 + (y_3 - y_2)^2}} \quad \cos \beta = \frac{y_3 - y_2}{\sqrt{(x_2 - x_3)^2 + (y_3 - y_2)^2}}$$

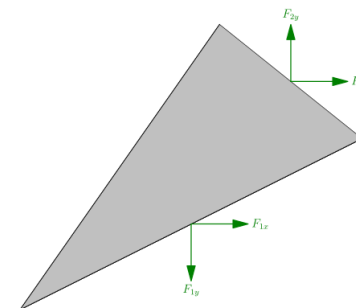


Solución

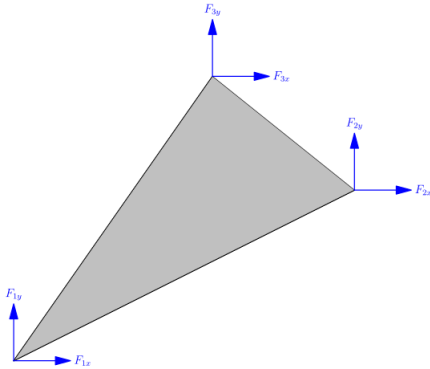


$$F_{1x} = q \sin \alpha t (y_2 - y_1) \quad F_{1y} = -q \cos \alpha t (x_2 - x_1)$$

$$F_{2x} = q \cos \beta t (y_3 - y_2) \quad F_{2y} = q \sin \beta t (x_2 - x_3)$$



$$\begin{aligned} F_{1x} &= \frac{F_{1x}}{2} & F_{1y} &= \frac{F_{1y}}{2} \\ F_{2x} &= \frac{F_{1x} + F_{2x}}{2} & F_{2y} &= \frac{F_{1y} + F_{2y}}{2} \\ F_{3x} &= \frac{F_{2x}}{2} & F_{3y} &= \frac{F_{2y}}{2} \end{aligned}$$



2. Calcular la integral mediante la cuadratura de Newton-Cotes para $n = 3$, los pesos w_i y los puntos de muestreo r_i

$$I = \int_1^5 x^3 e^x dx$$

Solución

$$k = n - 1 = 3 - 1 = 2$$

Calculando r_i

$$\begin{aligned} \int_{-1}^{+1} P(r) r^0 dr &= 0 \\ \int_{-1}^{+1} P(r) r^1 dr &= 0 \\ \int_{-1}^{+1} P(r) r^2 dr &= 0 \end{aligned}$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)(r - r_3)$$

Reemplazando

$$\begin{aligned} \int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3) dr &= 0 \\ \int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)r dr &= 0 \\ \int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)r^2 dr &= 0 \end{aligned}$$

Integrando

$$\begin{aligned} &\left[\frac{1}{4}r^4 - \frac{r_1 + r_2 + r_3}{3}r^3 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{2}r^2 - r_1r_2r_3r \right]_{-1}^{+1} \\ &= -\frac{2}{3}(3r_1r_2r_3 + r_1 + r_2 + r_3) \\ &\left[\frac{1}{5}r^5 - \frac{r_1 + r_2 + r_3}{4}r^4 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{3}r^3 - \frac{r_1r_2r_3}{2}r^2 \right]_{-1}^{+1} \\ &= \frac{2}{15}(5r_1r_2 + 5r_1r_3 + 5r_2r_3 + 3) \\ &\left[\frac{1}{6}r^6 - \frac{r_1 + r_2 + r_3}{5}r^5 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{4}r^4 - \frac{r_1r_2r_3}{3}r^3 \right]_{-1}^{+1} \\ &= -\frac{2}{15}(5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3) \end{aligned}$$

Formando el sistema de ecuaciones

$$\begin{aligned} 3r_1r_2r_3 + r_1 + r_2 + r_3 &= 0 \\ 5r_1r_2 + 5r_1r_3 + 5r_2r_3 &= -3 \\ 5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3 &= 0 \end{aligned}$$

Resolviendo

$$\begin{aligned} r_1 &= -\sqrt{\frac{3}{5}} \\ r_2 &= 0 \\ r_3 &= \sqrt{\frac{3}{5}} \end{aligned}$$

Calculando w_i

$$\begin{aligned} w_1 &= \int_{-1}^{+1} \frac{r-r_2}{r_1-r_2} \cdot \frac{r-r_3}{r_1-r_3} dr \\ w_2 &= \int_{-1}^{+1} \frac{r-r_1}{r_2-r_1} \cdot \frac{r-r_3}{r_2-r_3} dr \\ w_3 &= \int_{-1}^{+1} \frac{r-r_2}{r_3-r_2} \cdot \frac{r-r_1}{r_3-r_1} dr \end{aligned}$$

Reemplazando e integrando

$$\begin{aligned} w_1 &= \int_{-1}^{+1} \frac{r-0}{-\sqrt{\frac{3}{5}}-0} \cdot \frac{r-\sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}}} dr = \left(\frac{5}{18}r^3 - \frac{\sqrt{15}}{12}r^2 \right) \Big|_{-1}^{+1} = \frac{5}{9} \\ w_2 &= \int_{-1}^{+1} \frac{r+\sqrt{\frac{3}{5}}}{0+\sqrt{\frac{3}{5}}} \cdot \frac{r-\sqrt{\frac{3}{5}}}{0-\sqrt{\frac{3}{5}}} dr = \left(-\frac{5}{9}r^3 + r \right) \Big|_{-1}^{+1} = \frac{8}{9} \\ w_3 &= \int_{-1}^{+1} \frac{r-0}{\sqrt{\frac{3}{5}}-0} \cdot \frac{r+\sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}}} dr = \left(\frac{5}{18}r^3 + \frac{\sqrt{15}}{12}r^2 \right) \Big|_{-1}^{+1} = \frac{5}{9} \end{aligned}$$

Usando la fórmula

$$I = w'_1 f(r'_1) + w'_2 f(r'_2) + w'_3 f(r'_3)$$

Puntos de muestreo

$$\begin{aligned} r'_1 &= \frac{b+a}{2} + \frac{b-a}{2}r_1 = \frac{5+1}{2} + \frac{5-1}{2} \left(-\sqrt{\frac{3}{5}} \right) = 1.45081 \\ r'_2 &= \frac{b+a}{2} + \frac{b-a}{2}r_2 = \frac{5+1}{2} + \frac{5-1}{2}(0) = 3 \\ r'_3 &= \frac{b+a}{2} + \frac{b-a}{2}r_3 = \frac{5+1}{2} + \frac{5-1}{2} \left(\sqrt{\frac{3}{5}} \right) = 4.54919 \end{aligned}$$

Pesos

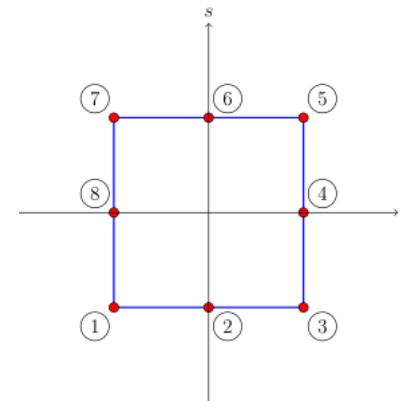
$$\begin{aligned} w'_1 &= \frac{b-a}{2}w_1 = \frac{5-1}{2} \left(\frac{5}{9} \right) = 1.11111 \\ w'_2 &= \frac{b-a}{2}w_2 = \frac{5-1}{2} \left(\frac{8}{9} \right) = 1.77778 \\ w'_3 &= \frac{b-a}{2}w_3 = \frac{5-1}{2} \left(\frac{5}{9} \right) = 1.11111 \end{aligned}$$

Reemplazando

$$\begin{aligned} I &= 1.11111 \left(1.45081^3 e^{1.45081} \right) + 1.77778 \left(3^3 e^3 \right) + 1.11111 \left(4.54919^3 e^{4.54919} \right) \\ &= 11819.38 \end{aligned}$$

3. Calcular las funciones de forma \mathbf{N}

- Elemento bidimensional
- Elemento unidimensional formado por los nodos ③-④-⑤



Solución

a) Elemento bidimensional

Coordenadas de los nodos

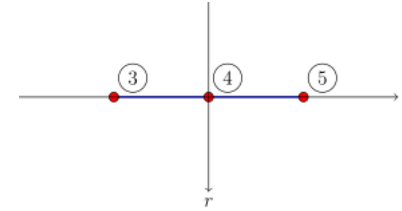
$$\begin{aligned} \textcircled{1} &= [r_1, s_1] = [-1, -1] & \textcircled{5} &= [r_5, s_5] = [1, 1] \\ \textcircled{2} &= [r_2, s_2] = [0, -1] & \textcircled{6} &= [r_6, s_6] = [0, 1] \\ \textcircled{3} &= [r_3, s_3] = [1, -1] & \textcircled{7} &= [r_7, s_7] = [-1, -1] \\ \textcircled{4} &= [r_4, s_4] = [1, 0] & \textcircled{8} &= [r_8, s_8] = [-1, 0] \end{aligned}$$

Reemplazando valores

$$\begin{aligned} N_1 &= \frac{r-r_2}{r_1-r_2} \cdot \frac{r-r_3}{r_1-r_3} \cdot \frac{s-s_8}{s_1-s_8} \cdot \frac{s-s_7}{s_1-s_7} \\ &= \frac{r-0}{-1-0} \cdot \frac{r-1}{-1-1} \cdot \frac{s-0}{-1-0} \cdot \frac{s-1}{-1-1} = \frac{1}{4}r(r-1)s(s-1) \\ N_2 &= \frac{r-r_3}{r_2-r_3} \cdot \frac{r-r_1}{r_2-r_1} \cdot \frac{s-s_6}{s_2-s_6} \\ &= \frac{r-1}{0-1} \cdot \frac{r-(-1)}{0-(-1)} \cdot \frac{s-1}{-1-1} = \frac{1}{2}(r-1)(r+1)(s-1) \\ N_3 &= \frac{r-r_2}{r_3-r_2} \cdot \frac{r-r_1}{r_3-r_1} \cdot \frac{s-s_4}{s_3-s_4} \cdot \frac{s-s_5}{s_3-s_5} \\ &= \frac{r-0}{1-0} \cdot \frac{r-(-1)}{1-(-1)} \cdot \frac{s-0}{-1-0} \cdot \frac{s-1}{-1-1} = \frac{1}{4}r(r+1)s(s-1) \\ N_4 &= \frac{r-r_8}{r_4-r_8} \cdot \frac{s-s_3}{s_4-s_3} \cdot \frac{s-s_5}{s_4-s_5} \\ &= \frac{r-(-1)}{1-(-1)} \cdot \frac{s-(-1)}{0-(-1)} \cdot \frac{s-1}{0-1} = -\frac{1}{2}(r+1)(s+1)(s-1) \\ N_5 &= \frac{r-r_4}{r_5-r_4} \cdot \frac{r-r_3}{r_5-r_3} \cdot \frac{s-s_6}{s_5-s_6} \cdot \frac{s-s_7}{s_5-s_7} \\ &= \frac{r-0}{1-0} \cdot \frac{r-(-1)}{1-(-1)} \cdot \frac{s-0}{1-0} \cdot \frac{s-(-1)}{1-(-1)} = \frac{1}{4}r(r+1)s(s+1) \end{aligned}$$

$$\begin{aligned} N_6 &= \frac{r-r_5}{r_6-r_5} \cdot \frac{r-r_7}{r_6-r_7} \cdot \frac{s-s_2}{s_6-s_2} \\ &= \frac{r-1}{0-1} \cdot \frac{r-(-1)}{0-(-1)} \cdot \frac{s-(-1)}{1-(-1)} = -\frac{1}{2}(r-1)(r+1)(s+1) \\ N_7 &= \frac{r-r_6}{r_7-r_6} \cdot \frac{r-r_5}{r_7-r_5} \cdot \frac{s-s_8}{s_7-s_8} \cdot \frac{s-s_1}{s_7-s_1} \\ &= \frac{r-0}{-1-0} \cdot \frac{r-1}{-1-1} \cdot \frac{s-0}{1-0} \cdot \frac{s-(-1)}{1-(-1)} = \frac{1}{4}r(r-1)s(s+1) \\ N_8 &= \frac{r-r_4}{r_8-r_4} \cdot \frac{s-s_1}{s_8-s_1} \cdot \frac{s-s_7}{s_8-s_7} \\ &= \frac{r-1}{-1-1} \cdot \frac{s-(-1)}{0-(-1)} \cdot \frac{s-1}{0-1} = \frac{1}{2}(r-1)(s+1)(s-1) \end{aligned}$$

b) Elemento unidimensional



$$\begin{aligned} \textcircled{3} &= r_3 = -1 & \textcircled{5} &= r_5 = 1 \\ \textcircled{4} &= r_4 = 0 \end{aligned}$$

Reemplazando valores

$$\begin{aligned} N_1 &= \frac{s-s_4}{s_3-s_4} \cdot \frac{s-s_5}{s_3-s_5} = \frac{s-0}{-1-0} \cdot \frac{s-1}{-1-1} = \frac{1}{2}s(s-1) \\ N_2 &= \frac{s-s_3}{s_4-s_3} \cdot \frac{s-s_5}{s_4-s_5} = \frac{s-(-1)}{0-(-1)} \cdot \frac{s-1}{0-1} = -(s^2-1) \\ N_3 &= \frac{s-s_4}{s_5-s_4} \cdot \frac{s-s_3}{s_5-s_3} = \frac{s-0}{1-0} \cdot \frac{s-(-1)}{1-(-1)} = \frac{1}{2}s(s+1) \end{aligned}$$