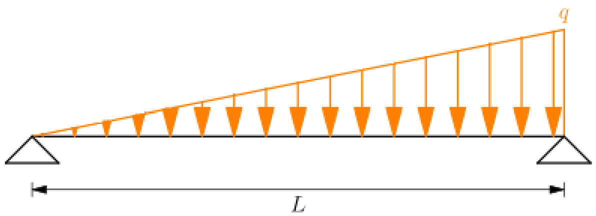
Introducción a elementos finitos
Segundo Parcial I-2016

1. Resolver la estructura con E, I, A constantes por el método de Ritz



Solución

La solución exacta es un polinomio de quinto grado, la aproximación del campo de desplazamientos será

$$v(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$$

Reemplazando $v(0) = 0$ y $v(L) = 0$

$$\alpha_0 + \alpha_1(0) + \alpha_2(0)^2 + \alpha_3(0)^3 + \alpha_4(0)^4 + \alpha_5(0)^5 = 0$$
$$\alpha_0 + \alpha_1(L) + \alpha_2(L)^2 + \alpha_3(L)^3 + \alpha_4(L)^4 + \alpha_5(L)^5 = 0$$

Resolviendo

$$\alpha_0 = 0$$
$$\alpha_1 = -(\alpha_2 L + \alpha_3 L^2 + \alpha_4 L^3 + \alpha_5 L^4)$$

Reemplazando en el campo de desplazamientos

$$v = -(\alpha_2 L + \alpha_3 L^2 + \alpha_4 L^3 + \alpha_5 L^4)x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$$

El momento es

$$M = E\,I\,\frac{d^2 v}{dx^2}$$

La curvatura es

$$\frac{d^2 v}{dx^2} = 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 + 20\alpha_5 x^3$$

La carga en la viga es

$$f = \frac{q}{L}x$$

El funcional de energía es

$$\pi = \int_0^L \frac{M^2}{2EI}\,dx - \int_0^L f\,v\,dx$$

Reemplazando

$$\pi = \int_0^L \frac{EI}{2}\left(2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 + 20\alpha_5 x^3\right)^2 dx - \int_0^L \frac{q}{L}x\,[-(\alpha_2 L + \alpha_3 L^2 + \alpha_4 L^3 + \alpha_5 L^4)x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5]\,dx$$

Integrando

$$\pi = \frac{qL^3}{12}\,\alpha_2 + \frac{2qL^4}{15}\,\alpha_3 + \frac{qL^5}{6}\,\alpha_4 + \frac{4qL^6}{21}\,\alpha_5 + 2EIL\,\alpha_2^2 + 6EIL^2\,\alpha_2\,\alpha_3 + 6EIL^3\,\alpha_3^2 + 8EIL^3\,\alpha_2\,\alpha_4 + 18EIL^4\,\alpha_3\,\alpha_4 + 10EIL^4\,\alpha_2\,\alpha_5 + \frac{72EIL^5}{5}\,\alpha_4^2 + 24EIL^5\,\alpha_3\,\alpha_5 + 40EIL^6\,\alpha_4\,\alpha_5 + \frac{200EIL^7}{7}\,\alpha_5^2$$

Minimizando el funcional

$$\frac{\partial\pi}{\partial\alpha_2} = 4EIL\,\alpha_2 + 6EIL^2\,\alpha_3 + 8EIL^3\,\alpha_4 + 10EIL^4\,\alpha_5 + \frac{qL^3}{12} = 0$$
$$\frac{\partial\pi}{\partial\alpha_3} = 6EIL^2\,\alpha_2 + 12EIL^3\,\alpha_3 + 18EIL^4\,\alpha_4 + 24EIL^5\,\alpha_5 + \frac{2qL^4}{15} = 0$$
$$\frac{\partial\pi}{\partial\alpha_4} = 8EIL^3\,\alpha_2 + 18EIL^4\,\alpha_3 + \frac{144EIL^5}{5}\,\alpha_4 + 40EIL^6\,\alpha_5 + \frac{qL^5}{6} = 0$$
$$\frac{\partial\pi}{\partial\alpha_5} = 10EIL^4\,\alpha_2 + 24EIL^5\,\alpha_3 + 40EIL^6\,\alpha_4 + \frac{400EIL^7}{7}\,\alpha_5 + \frac{4qL^6}{21} = 0$$

Formando el sistema de ecuaciones

$$\begin{aligned}
4EIL\,\alpha_2 + 6EIL^2\,\alpha_3 + 8EIL^3\,\alpha_4 + 10EIL^4\,\alpha_5 &= -\frac{qL^3}{12} \\
6EIL^2\,\alpha_2 + 12EIL^3\,\alpha_3 + 18EIL^4\,\alpha_4 + 24EIL^5\,\alpha_5 &= -\frac{2qL^4}{15} \\
8EIL^3\,\alpha_2 + 18EIL^4\,\alpha_3 + \frac{144EIL^5}{5}\,\alpha_4 + 40EIL^6\,\alpha_5 &= -\frac{qL^5}{6} \\
10EIL^4\,\alpha_2 + 24EIL^5\,\alpha_3 + 40EIL^6\,\alpha_4 + \frac{400EIL^7}{7}\,\alpha_5 &= -\frac{4qL^6}{21}
\end{aligned}$$

Resolviendo

$$\begin{aligned}
\alpha_2 &= 0 \\
\alpha_3 &= -\frac{qL}{36EI} \\
\alpha_4 &= 0 \\
\alpha_5 &= \frac{q}{120EIL}
\end{aligned}$$

Reemplazando en α_1

$$\alpha_1 = \frac{7qL^3}{360EI}$$

Reemplazando en v

$$v = \frac{q}{360EIL}\left(7L^4x - 10L^2x^3 + 3x^5\right)$$

2. Integrar

$$\int u\,\frac{d^4v}{dx^4}\,dx$$

Solución

Para la solución se usará el teorema de Gauss

$$\int a\,\frac{\partial b}{\partial x_i}\,dx = ab - \int b\,\frac{\partial a}{\partial x_i}\,dx + c$$

Integrando

$$\int u\,\frac{d^4v}{dx^4}\,dx = u\,\frac{d^3v}{dx^3} - \int \frac{d^3v}{dx^3}\,\frac{du}{dx}\,dx$$

Integrando el segundo término

$$\int \frac{du}{dx}\,\frac{d^3v}{dx^3}\,dx = \frac{du}{dx}\,\frac{d^2v}{dx^2} - \int \frac{d^2v}{dx^2}\,\frac{d^2u}{dx^2}\,dx$$

Reemplazando

$$\int u\,\frac{d^4v}{dx^4}\,dx = u\,\frac{d^3v}{dx^3} - \left(\frac{du}{dx}\,\frac{d^2v}{dx^2} - \int \frac{d^2u}{dx^2}\,\frac{d^2v}{dx^2}\,dx\right)$$

Simplificando

$$\int u\,\frac{d^4v}{dx^4}\,dx = u\,\frac{d^3v}{dx^3} - \frac{du}{dx}\,\frac{d^2v}{dx^2} + \int \frac{d^2u}{dx^2}\,\frac{d^2v}{dx^2}\,dx + c$$

3. Defina trabajo virtual

Solución

Es el trabajo que realiza la fuerza $\boldsymbol{F}$ debido a un desplazamiento virtual $\delta\boldsymbol{r}$

$$\delta W = \boldsymbol{F} \cdot \delta\boldsymbol{r}$$