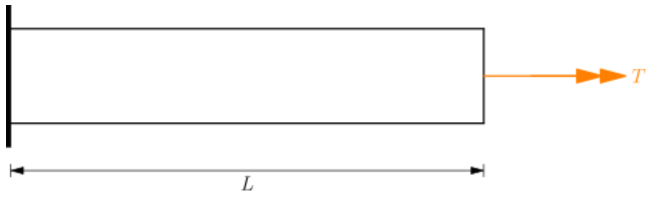
Introducción a elementos finitos
Examen final I-2016

1. Calcular la matriz de rigidez $\boldsymbol{K}$ mediante el método de balance de energía



Solución

Deformación de la barra

$$\phi = \frac{TL}{GI_p}$$

Despejando T

$$T = \frac{GI_p}{L}\,\phi$$

Energía de deformación por torsión

$$U_i = \frac{1}{2}\,T\,\phi\,dx$$

Reemplazando T en U_i

$$U_i = \frac{1}{2}\,\frac{GI_p}{L}\,\phi^2$$

La deformación unitaria es

$$\theta = \frac{\phi}{L}$$

Despejando ϕ

$$\phi = L\,\theta$$

Reemplazando ϕ en U_i

$$U_i = \frac{1}{2}\,GI_pL\,\theta^2$$

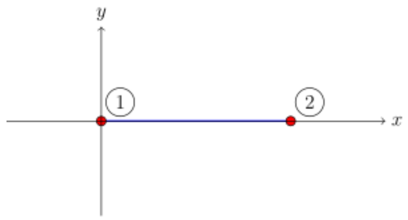
Para condiciones variables

$$U_i = \int \frac{1}{2}\,GI_p\,\theta^2\,dx$$

Funcional de energía

$$\pi = \int_0^L \frac{1}{2}\,GI_p\,\theta^2\,dx - T(\theta_2 - \theta_1) = \int_0^L \frac{1}{2}\,\theta^{\mathrm{T}}\,GI_p\,\theta\,dx - \sum_{i=1}^{2}\theta_i\,T_i$$

Usando un elemento de dos nodos



Aproximación del campo de desplazamientos

$$\phi = \alpha_0 + \alpha_1 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

Reemplazando $\phi(0) = \phi_1$ y $\phi(L) = \phi_2$

$$\begin{aligned} \alpha_0 + \alpha_1(0) &= \phi_1 \\ \alpha_0 + \alpha_1(L) &= \phi_2 \end{aligned}$$

Simplificando

$$\begin{aligned} \alpha_0 &= \phi_1 \\ \alpha_0 + L\alpha_1 &= \phi_2 \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Resolviendo

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Reemplazando

$$\phi = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{L}x & \frac{1}{L}x \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \boldsymbol{N}\, \boldsymbol{\phi}_i$$

Deformación angular

$$\theta = \frac{d\phi}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \boldsymbol{B}\, \boldsymbol{\phi}_i$$

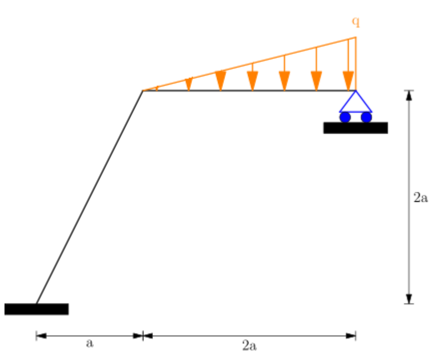
Matriz de rigidez

$$\boldsymbol{K} = \int_0^L \boldsymbol{B}^{\mathrm{T}}\, GI_p\, \boldsymbol{B}\, dx$$

Reemplazando e integrando

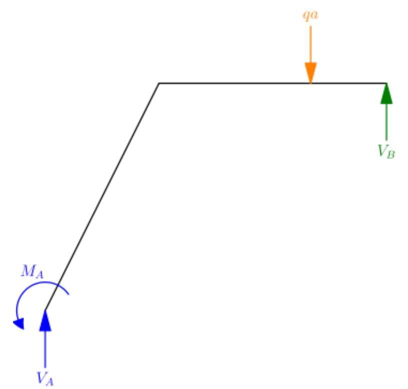
$$\boldsymbol{K} = \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} GI_p \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx = \frac{GI_p}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2. Resolver la estructura por cualquier método



Solución

Estructura equivalente



Sumatoria de fuerzas y momentos

$$\begin{aligned} V_A + V_B - qa &= 0 \\ M_A - qa\left(\frac{7}{3}\,a\right) + V_B(3a) &= 0 \\ qa\left(\frac{2}{3}\,a\right) - V_A(3a) &= 0 \end{aligned}$$

Resolviendo

$$\begin{aligned} V_A &= \frac{2}{9}\,qa \\ M_A &= \frac{7}{6}\,qa^2 \\ V_B &= \frac{7}{9}\,qa \end{aligned}$$

3. Calcular la integral mediante la cuadratura de Newton-Cotes para $n = 2$, los pesos w_i y los puntos de muestreo r_i

$$I = \int_0^2 x^2 - 5x\, dx$$

Solución

Número de términos

$$k = n - 1 = 2 - 1 = 1$$

Calculando r_i

$$\int_{-1}^{+1} P(r)\ r^0\, dr = 0$$
$$\int_{-1}^{+1} P(r)\ r^1\, dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2)\, dr = 0$$
$$\int_{-1}^{+1} (r - r_1)(r - r_2) r\, dr = 0$$

Integrando

$$\left(\frac{1}{3}r^3 - \frac{r_1 + r_2}{2}r^2 + r_1 r_2 r\right)\Bigg|_{-1}^{+1} = 2\left(r_1 r_2 + \frac{1}{3}\right)$$
$$\left(\frac{1}{4}r^4 - \frac{r_1 + r_2}{3}r^3 + \frac{r_1 r_2}{2}r^2\right)\Bigg|_{-1}^{+1} = -\frac{2}{3}(r_1 + r_2)$$

Formando el sistema de ecuaciones

$$r_1 r_2 = -\frac{1}{3}$$
$$r_1 + r_2 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{1}{3}}$$
$$r_2 = \sqrt{\frac{1}{3}}$$

Calculando w_i

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2}\, dr$$
$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1}\, dr$$

Reemplazando e integrando

$$w_1 = \int_{-1}^{+1} \frac{r - \sqrt{\frac{1}{3}}}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}}\, dr = \left(-\frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r\right)\Bigg|_{-1}^{+1} = 1$$
$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}}\, dr = \left(\frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r\right)\Bigg|_{-1}^{+1} = 1$$

Usando la fórmula

$$I = w_1' f(r_1') + w_2' f(r_2')$$

Puntos de muestreo

$$r_1' = \frac{b + a}{2} + \frac{b - a}{2}r_1 = \frac{2 + 0}{2} + \frac{2 - 0}{2}\left(-\sqrt{\frac{1}{3}}\right) = 0.42265$$
$$r_2' = \frac{b + a}{2} + \frac{b - a}{2}r_2 = \frac{2 + 0}{2} + \frac{2 - 0}{2}\left(\sqrt{\frac{1}{3}}\right) = 1.57735$$

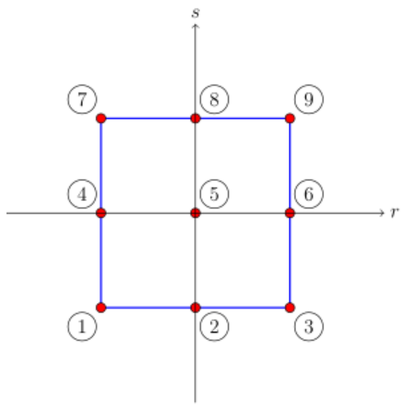
Pesos

$$w_1' = \frac{b - a}{2}w_1 = \frac{2 - 0}{2}(1) = 1$$
$$w_2' = \frac{b - a}{2}w_2 = \frac{2 - 0}{2}(1) = 1$$

Reemplazando

$$I = 1\left[0.42265^2 - 5(0.42265)\right] + 1\left[1.57735^2 - 5(1.57735)\right] = -7.33333$$

4. Calcular las funciones de forma $\boldsymbol{N}$



Solución

Coordenadas de los nodos

$$\begin{aligned}
&① = [r_1, s_1] = [-1, -1] \quad ⑥ = [r_6, s_6] = [1, 0]\\
&② = [r_2, s_2] = [0, -1] \quad ⑦ = [r_7, s_7] = [-1, 1]\\
&③ = [r_3, s_3] = [1, -1] \quad ⑧ = [r_8, s_8] = [0, 1]\\
&④ = [r_4, s_4] = [-1, 0] \quad ⑨ = [r_9, s_9] = [1, 1]\\
&⑤ = [r_4, s_4] = [0, 0]
\end{aligned}$$

Reemplazando valores

$$N_1 = \frac{r - r_2}{r_1 - r_2} \cdot \frac{r - r_3}{r_1 - r_3} \cdot \frac{s - s_4}{s_1 - s_4} \cdot \frac{s - s_7}{s_1 - s_7} = \frac{r - 0}{-1 - 0} \cdot \frac{r - 1}{-1 - 1} \cdot \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1}$$
$$= \frac{1}{4} r(r - 1)s(s - 1)$$

$$N_2 = \frac{r - r_1}{r_2 - r_1} \cdot \frac{r - r_3}{r_2 - r_3} \cdot \frac{s - s_5}{s_2 - s_5} \cdot \frac{s - s_8}{s_2 - s_8} = \frac{r - (-1)}{0 - (-1)} \cdot \frac{r - 1}{0 - 1} \cdot \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1}$$
$$= -\frac{1}{2} r(r + 1)s(s - 1)$$

$$N_3 = \frac{r - r_1}{r_3 - r_1} \cdot \frac{r - r_2}{r_3 - r_2} \cdot \frac{s - s_6}{s_3 - s_6} \cdot \frac{s - s_9}{s_3 - s_9} = \frac{r - (-1)}{1 - (-1)} \cdot \frac{r - 0}{1 - 0} \cdot \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1}$$
$$= \frac{1}{4} r(r + 1)s(s - 1)$$

$$N_4 = \frac{r - r_5}{r_4 - r_5} \cdot \frac{r - r_6}{r_4 - r_6} \cdot \frac{s - s_1}{s_4 - s_1} \cdot \frac{s - s_7}{s_4 - s_7} = \frac{r - 0}{-1 - 0} \cdot \frac{r - 1}{-1 - 1} \cdot \frac{s - (-1)}{0 - (-1)} \cdot \frac{s - 1}{0 - 1}$$
$$= -\frac{1}{2} r(r - 1)(s + 1)(s - 1)$$

$$N_5 = \frac{r - r_4}{r_5 - r_4} \cdot \frac{r - r_6}{r_5 - r_6} \cdot \frac{s - s_2}{s_5 - s_2} \cdot \frac{s - s_8}{s_5 - s_8} = \frac{r - (-1)}{0 - (-1)} \cdot \frac{r - 1}{0 - 1} \cdot \frac{s - (-1)}{0 - (-1)} \cdot \frac{s - 1}{0 - 1}$$
$$= (r + 1)(r - 1)(s + 1)s(s - 1)$$

$$N_6 = \frac{r - r_4}{r_6 - r_4} \cdot \frac{r - r_5}{r_6 - r_5} \cdot \frac{s - s_3}{s_6 - s_3} \cdot \frac{s - s_9}{s_6 - s_9} = \frac{r - (-1)}{1 - (-1)} \cdot \frac{r - 0}{1 - 0} \cdot \frac{s - (-1)}{0 - (-1)} \cdot \frac{s - 1}{0 - 1}$$
$$= -\frac{1}{2} r(r - 1)(s + 1)(s - 1)$$

$$N_7 = \frac{r - r_8}{r_7 - r_8} \cdot \frac{r - r_9}{r_7 - r_9} \cdot \frac{s - s_1}{s_7 - s_1} \cdot \frac{s - s_4}{s_7 - s_4} = \frac{r - 0}{-1 - 0} \cdot \frac{r - 1}{-1 - 1} \cdot \frac{s - (-1)}{1 - (-1)} \cdot \frac{s - 0}{1 - 0}$$
$$= \frac{1}{4} r(r - 1)s(s + 1)$$

$$N_8 = \frac{r - r_7}{r_8 - r_7} \cdot \frac{r - r_9}{r_8 - r_9} \cdot \frac{s - s_2}{s_8 - s_2} \cdot \frac{s - s_5}{s_8 - s_5} = \frac{r - (-1)}{0 - (-1)} \cdot \frac{r - 1}{0 - 1} \cdot \frac{s - (-1)}{1 - (-1)} \cdot \frac{s - 0}{1 - 0}$$
$$= -\frac{1}{2} (r + 1)(r - 1)s(s + 1)$$

$$N_9 = \frac{r - r_7}{r_9 - r_7} \cdot \frac{r - r_8}{r_9 - r_8} \cdot \frac{s - s_3}{s_9 - s_3} \cdot \frac{s - s_6}{s_9 - s_6} = \frac{r - (-1)}{1 - (-1)} \cdot \frac{r - 0}{1 - 0} \cdot \frac{s - (-1)}{1 - (-1)} \cdot \frac{s - 0}{1 - 0}$$
$$= \frac{1}{4} r(r + 1)s(s + 1)$$