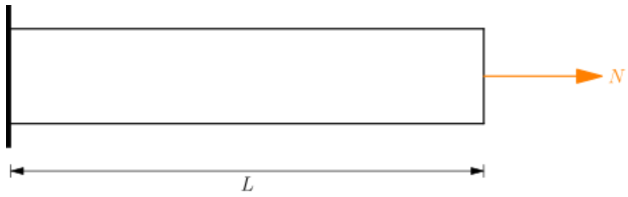
Introducción a elementos finitos
Examen final II-2016 Segunda opción

1. Obtener la matriz de rigidez mediante el método de Galerkin



Solución 1

Ecuación diferencial

$$EA\,\frac{d^2u(x)}{dx^2} + q(x) = 0$$

No hay carga variable

$$EA\,\frac{d^2u(x)}{dx^2} = 0$$

La aproximación de los desplazamientos será

$$u(x) = \phi(x)$$

La función ponderada será

$$W = \phi_i(x)$$

Aplicando el método de Galerkin

$$\int_0^L \left(EA\frac{d^2\phi}{dx^2}\right)\phi_i\,dx = \int_0^L \phi_i\,EA\frac{d^2\phi}{dx^2}\,dx = 0$$

Usando el teorema de Gauss o integrando por partes

$$\left.\left(\phi_i\,EA\frac{d\phi}{dx}\right)\right|_0^L - \int_0^L \frac{d\phi_i}{dx}EA\frac{d\phi}{dx}\,dx = 0$$

Reordenando

$$\int_0^L \frac{d\phi_i}{dx}EA\frac{d\phi}{dx}\,dx = \left.\left(\phi_i\,EA\frac{d\phi}{dx}\right)\right|_0^L$$

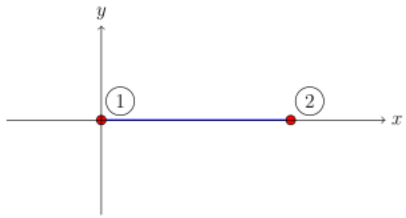
Reemplazando $F = EA\frac{d\phi}{dx}$

$$\int_0^L \frac{d\phi_i}{dx}EA\frac{d\phi}{dx}\,dx = \left.(\phi_i\,F)\right|_0^L$$

La rigidez es

$$K = EA\int_0^L \frac{d\phi_i}{dx}\,\frac{d\phi}{dx}\,dx$$

Usando un elemento de dos nodos



Aproximación del campo de desplazamientos

$$\phi = \alpha_0 + \alpha_1 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

Reemplazando $\phi(0) = \phi_1$ y $\phi(L) = \phi_2$

$$\begin{aligned} \alpha_0 + \alpha_1(0) &= \phi_1 \\ \alpha_0 + \alpha_1(L) &= \phi_2 \end{aligned}$$

Simplificando

$$\begin{aligned} \alpha_0 &= \phi_1 \\ \alpha_0 + L\alpha_1 &= \phi_2 \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Resolviendo

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Reemplazando

$$\phi = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{L}x & \frac{1}{L}x \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Deformación unitaria

$$\varepsilon = \frac{d\phi}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = -\frac{1}{L}\phi_1 + \frac{1}{L}\phi_2$$

Reemplazando $\frac{d\phi_1}{dx} = -\frac{1}{L}$

$$EA \int_0^L \frac{d\phi_1}{dx} \frac{d\phi}{dx} \, dx = EA \int_0^L -\frac{1}{L} \left(-\frac{1}{L}\phi_1 + \frac{1}{L}\phi_2 \right) dx$$

Reemplazando $\frac{d\phi_2}{dx} = \frac{1}{L}$

$$EA \int_0^L \frac{d\phi_2}{dx} \frac{d\phi}{dx} \, dx = EA \int_0^L \frac{1}{L} \left(-\frac{1}{L}\phi_1 + \frac{1}{L}\phi_2 \right) dx$$

Formando un sistema de ecuaciones

$$\frac{EA}{L^2} \int_0^L \phi_1 - \phi_2 \, dx = F_1$$
$$\frac{EA}{L^2} \int_0^L -\phi_1 + \phi_2 \, dx = F_2$$

En forma matricial

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

La matriz de rigidez es

$$K = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2. Integrar usando la fórmula de Newton-Cotes para $n = 2$

$$I = \int_{-1}^{+1} \int_{-1}^{+1} e^5 \, r \, ds \, dr$$

Solución

Pesos y puntos de muestreo

$$\begin{aligned} w_1 &= 1 \quad r_1 = -\sqrt{\frac{1}{3}} \quad s_1 = -\sqrt{\frac{1}{3}} \\ w_2 &= 1 \quad r_2 = \sqrt{\frac{1}{3}} \quad s_2 = \sqrt{\frac{1}{3}} \end{aligned}$$

Reordenando

$$I = \int_{-1}^{+1} \int_{-1}^{+1} e^5 \, r \, ds \, dr = \int_{-1}^{+1} dr \int_{-1}^{+1} e^5 \, r \, ds = \sum_{i=1}^{2} \sum_{j=1}^{2} w_i \, w_j \, f(r_i, s_j)$$

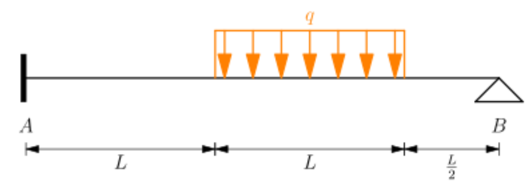
Usando la fórmula

$$\begin{aligned} I &= \sum_{i=1}^{2} \sum_{j=1}^{2} w_i \, w_j \, f(r_i) \\ &= w_1 \, w_1 \, f(r_1) + w_1 \, w_2 \, f(r_1) + w_2 \, w_1 \, f(r_2) + w_2 \, w_2 \, f(r_2) \end{aligned}$$

Reemplazando

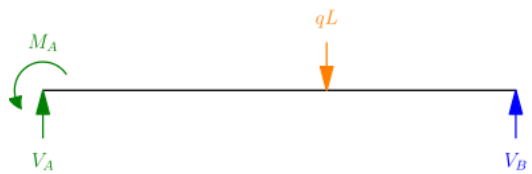
$$\begin{aligned} I &= 1 \cdot 1 \cdot e^5 \left(-\sqrt{\frac{1}{3}} \right) + 1 \cdot 1 \cdot e^5 \left(-\sqrt{\frac{1}{3}} \right) + 1 \cdot 1 \cdot e^5 \left(\sqrt{\frac{1}{3}} \right) + 1 \cdot 1 \cdot e^5 \left(\sqrt{\frac{1}{3}} \right) \\ &= 0 \end{aligned}$$

3. Resolver la estructura por cualquier método



Solución

Estructura equivalente

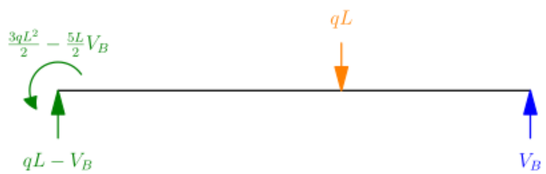


Suma de fuerzas y momentos

$$V_A - qL + V_B = 0$$
$$M_A - qL\left(\frac{3L}{2}\right) + V_B\left(\frac{5L}{2}\right) = 0$$

Despejando V_A y M_A

$$V_A = qL - V_B$$
$$M_A = \frac{3qL^2}{2} - \frac{5L}{2}V_B$$



Momento de $0 \leqslant x \leqslant L$

$$M = -M_A + V_A x = -\frac{3qL^2}{2} + \frac{5L}{2}V_B + (qL - V_B)x$$

Momento de $L \leqslant x \leqslant 2L$

$$M = -M_A + V_A x - \frac{q}{2}(x-L)^2 = -2qL^2 + \frac{5L}{2}V_B + 2qLx - V_B x - \frac{q}{2}x^2$$

Momento de $\frac{L}{2} \geqslant x \geqslant 0$

$$M = V_B x$$

Energía de deformación por flexión

$$U_i = \int_0^L \frac{M^2}{2EI}\,dx + \int_L^{2L} \frac{M^2}{2EI}\,dx + \int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dx$$

Reemplazando

$$\begin{aligned} U_i = &\frac{1}{2EI}\int_0^L \left[-\frac{3qL^2}{2} + \frac{5L}{2}V_B + (qL - V_B)x\right]^2 dx \\ &+ \frac{1}{2EI}\int_L^{2L} \left(-2qL^2 + \frac{5L}{2}V_B + 2qLx - V_B x - \frac{q}{2}x^2\right)^2 dx \\ &+ \frac{1}{2EI}\int_0^{\frac{L}{2}} (V_B x)^2 dx \end{aligned}$$

Integrando

$$U_i = \frac{L^3}{240EI}\left(136q^2L^2 - 550qLV_B + 625V_B^2\right)$$

Minimizando

$$\frac{dU_i}{dV_B} = -\frac{5L^3}{24EI}\left(11qL - 25V_B\right) = 0$$

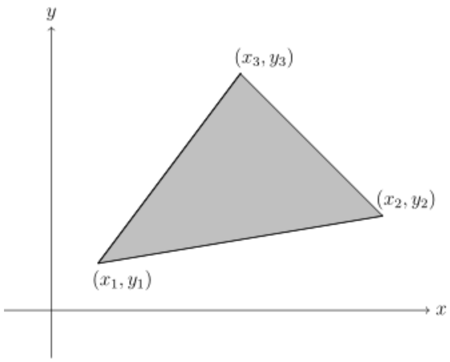
Despejando V_B

$$V_B = \frac{11qL}{25}$$

Reemplazando en las demás reacciones

$$V_A = qL - V_B = qL - \frac{11qL}{25} = \frac{14qL}{25}$$
$$M_A = \frac{3qL^2}{2} - \frac{5L}{2}V_B = \frac{3qL^2}{2} - \frac{5L}{2}\left(\frac{11qL}{25}\right) = \frac{2qL^2}{5}$$

4. Mediante el método directo hallar las constantes de los polinomios de interpolación para un triángulo de deformación constante



Solución

Campo de desplazamientos

$$\phi^{\mathrm{T}} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{bmatrix}$$

Funciones de aproximación

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v(x,y) = \alpha_4 + \alpha_5 x + \alpha_6 y$$

Reemplazando coordenadas nodales

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$
$$v_1 = \alpha_4 + \alpha_5 x_1 + \alpha_6 y_1$$
$$v_2 = \alpha_4 + \alpha_5 x_2 + \alpha_6 y_2$$
$$v_3 = \alpha_4 + \alpha_5 x_4 + \alpha_6 y_3$$

Para α_1

$$\alpha_1 = \frac{\begin{vmatrix} u_1 & x_1 & y_1 \\ u_2 & x_2 & y_2 \\ u_3 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \frac{(y_3x_2 - y_2x_3)u_1 + (y_1x_3 - y_3x_1)u_2 + (y_1x_2 - y_2x_1)u_3}{2A}$$

Realizando un cambio de variable

$$a_1 = y_3x_2 - y_2x_3$$
$$a_2 = y_1x_3 - y_3x_1$$
$$a_3 = y_1x_2 - y_2x_1$$

Reemplazando

$$\alpha_1 = \frac{1}{2A}(a_1u_1 + a_2u_2 + a_3u_3)$$

Para α_2

$$\alpha_2 = \frac{\begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \frac{(y_2 - y_3)u_1 + (y_3 - y_1)u_2 + (y_1 - y_2)u_3}{2A}$$

Realizando un cambio de variable

$$b_1 = y_2 - y_3$$
$$b_2 = y_3 - y_1$$
$$b_3 = y_1 - y_2$$

Reemplazando

$$\alpha_2 = \frac{1}{2A}(b_1u_1 + b_2u_2 + b_3u_3)$$

Para α_3

$$\alpha_3 = \frac{\begin{vmatrix} 1 & x_1 & u_1 \\ 1 & x_2 & u_2 \\ 1 & x_3 & u_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \frac{(x_3 - x_2)u_1 + (x_1 - x_3)u_2 + (x_2 - x_1)u_3}{2A}$$

Realizando un cambio de variable

$$c_1 = x_3 - x_2$$
$$c_2 = x_1 - x_3$$
$$c_3 = x_2 - x_1$$

Reemplazando

$$\alpha_3 = \frac{1}{2A}(c_1 u_1 + c_2 u_2 + c_3 u_3)$$

Para v las soluciones son iguales

$$\alpha_4 = \frac{1}{2A}(a_1 v_1 + a_2 v_2 + a_3 v_3)$$
$$\alpha_5 = \frac{1}{2A}(b_1 v_1 + b_2 v_2 + b_3 v_3)$$
$$\alpha_6 = \frac{1}{2A}(c_1 v_1 + c_2 v_2 + c_3 v_3)$$