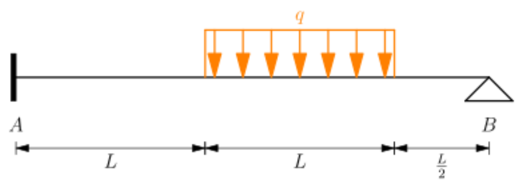
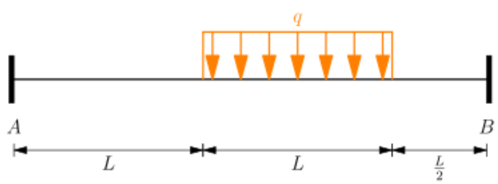
Introducción a elementos finitos
Tarea 2 I-2016

Resolver por el método de Castigliano



Solución 1

La estructura se dividirá en dos estructuras, las reacciones se obtendrán mediante superposición.
La primera estructura es



Momento de $0 \leqslant x \leqslant L$

$$M = -M_A + V_A x$$

Momento de $L \leqslant x \leqslant 2L$

$$\begin{aligned} M &= -M_A + V_A x - \frac{q}{2}(x-L)^2 \\ &= -\left(M_A + \frac{qL^2}{2}\right) + (V_A + qL)\,x - \frac{q}{2}x^2 \end{aligned}$$

Momento de $2L \leqslant x \leqslant \frac{5L}{2}$

$$\begin{aligned} M &= -M_A + V_A x - qL\left(x - \frac{3L}{2}\right) \\ &= -\left(M_A - \frac{3qL^2}{2}\right) + (V_A - qL)\,x \end{aligned}$$

El desplazamiento es cero en el punto A

$$\frac{\partial U_i}{\partial V_A} = 0$$
$$\frac{\partial U_i}{\partial M_A} = 0$$

Derivando U_i respecto de V_A

$$\begin{aligned} \frac{\partial U_i}{\partial V_A} &= \frac{1}{EI}\int_0^{\frac{5L}{2}} M\frac{\partial M}{\partial V_A}\ dx \\ &= \frac{1}{EI}\int_0^{L} (-M_A + V_A x)\,x\ dx \\ &+ \frac{1}{EI}\int_L^{2L} \left[-\left(M_A + \frac{qL^2}{2}\right) + (V_A + qL)\,x - \frac{q}{2}x^2\right] x\ dx \\ &+ \frac{1}{EI}\int_{2L}^{\frac{5L}{2}} \left[-\left(M_A - \frac{3qL^2}{2}\right) + (V_A - qL)\,x\right] x\ dx \end{aligned}$$

Derivando U_i respecto de M_A

$$\begin{aligned} \frac{\partial U_i}{\partial M_A} &= \frac{1}{EI}\int_0^{\frac{5L}{2}} M\frac{\partial M}{\partial M_A}\ dx \\ &= \frac{1}{EI}\int_0^{L} (-M_A + V_A x)\,(-1)\ dx \\ &+ \frac{1}{EI}\int_L^{2L} \left[-\left(M_A + \frac{qL^2}{2}\right) + (V_A + qL)\,x - \frac{q}{2}x^2\right] (-1)\ dx \\ &+ \frac{1}{EI}\int_{2L}^{\frac{5L}{2}} \left[-\left(M_A - \frac{3qL^2}{2}\right) + (V_A - qL)\,x\right] (-1)\ dx \end{aligned}$$

Multiplicando

$$\frac{1}{EI}\int_0^{L} -M_A x + V_A x^2\ dx + \frac{1}{EI}\int_L^{2L} -\left(M_A + \frac{qL^2}{2}\right)x + (V_A + qL)\,x^2$$
$$-\frac{q}{2}x^3\ dx + \frac{1}{EI}\int_{2L}^{\frac{5L}{2}} -\left(M_A - \frac{3qL^2}{2}\right)x + (V_A - qL)\,x^2\ dx = 0$$
$$\frac{1}{EI}\int_0^{L} M_A - V_A x\ dx + \frac{1}{EI}\int_L^{2L} \left(M_A + \frac{qL^2}{2}\right) - (V_A + qL)\,x$$
$$+\frac{q}{2}x^2\ dx + \frac{1}{EI}\int_{2L}^{\frac{5L}{2}} \left(M_A - \frac{3qL^2}{2}\right) - (V_A - qL)\,x\ dx = 0$$

Integrando

$$\frac{1}{EI}\left[-\frac{M_A}{2}x^2+\frac{V_A}{3}x^3\right]\Big|_0^L$$
$$+\frac{1}{EI}\left[-\frac{1}{2}\left(M_A+\frac{qL^2}{2}\right)x^2+\frac{1}{3}\left(V_A+qL\right)x^3-\frac{q}{8}x^4\right]\Big|_L^{2L}$$
$$+\frac{1}{EI}\left[-\frac{1}{2}\left(M_A-\frac{3qL^2}{2}\right)x^2+\frac{1}{3}\left(V_A-qL\right)x^3\right]\Big|_{2L}^{\frac{5L}{2}}=0$$
$$\frac{1}{EI}\left(M_Ax-\frac{V_A}{2}x^2\right)\Big|_0^L$$
$$+\frac{1}{EI}\left[\left(M_A+\frac{qL^2}{2}\right)x-\frac{1}{2}\left(V_A+qL\right)x^2+\frac{q}{6}x^3\right]\Big|_L^{2L}$$
$$+\frac{1}{EI}\left[\left(M_A-\frac{3qL^2}{2}\right)x-\frac{1}{2}\left(V_A-qL\right)x^2\right]\Big|_{2L}^{\frac{5L}{2}}=0$$

Simplificando

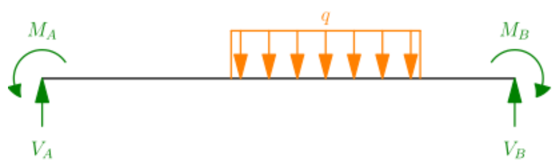
$$\frac{125L^3}{24}V_A-\frac{25L^2}{8}M_A=\frac{55qL^4}{48}$$
$$-\frac{25L^2}{8}V_A+\frac{5L}{2}M_A=-\frac{13qL^3}{24}$$

Resolviendo

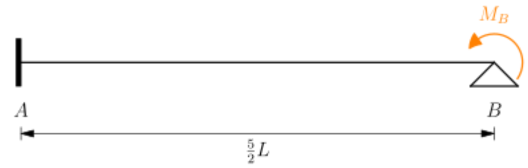
$$V_A=\frac{9qL}{25}$$
$$M_A=\frac{7qL^2}{30}$$

Por equilibrio las reacciones son

$$V_A=\frac{9qL}{25}\qquad V_B=\frac{16qL}{25}$$
$$M_A=\frac{7qL^2}{30}\qquad M_B=\frac{qL^2}{3}$$



La segunda estructura es



El momento de $0\leqslant x\leqslant\frac{5L}{2}$ es

$$M=-\frac{qL^2}{3}+V_Bx$$

El desplazamiento vertical es cero en el punto B

$$\frac{\partial U_i}{\partial V_B}=0$$

Derivando U_i respecto de V_A

$$\frac{\partial U_i}{\partial V_B}=\frac{1}{EI}\int_0^{\frac{5L}{2}}M\frac{\partial M}{\partial V_B}\ dx$$
$$=\frac{1}{EI}\int_0^{\frac{5L}{2}}\left(-\frac{qL^2}{3}+V_Bx\right)x\ dx$$

Multiplicando

$$\frac{1}{EI}\int_0^{\frac{5L}{2}}-\frac{qL^2}{3}x+V_Bx^2\ dx=0$$

Integrando

$$\frac{1}{EI}\left(-\frac{qL^2}{6}x^2+\frac{V_B}{3}x^3\right)\Big|_0^{\frac{5L}{2}}=0$$

Simplificando

$$\frac{1}{EI}\left(-\frac{25qL^4}{24}+\frac{125L^3}{24}V_B\right)=0$$

Resolviendo

$$V_B=\frac{qL}{5}$$

Por equilibrio las reacciones son

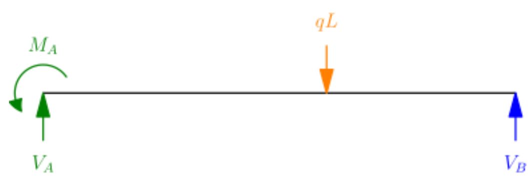
$$V_A = \frac{qL}{5} \qquad V_B = \frac{qL}{5}$$
$$M_A = \frac{qL^2}{6} \qquad M_B = \frac{qL^2}{3}$$

Por superposición las reacciones son

$$V_A = \frac{9qL}{25} + \frac{qL}{5} = \frac{14qL}{25}$$
$$M_A = \frac{7qL^2}{30} + \frac{qL^2}{6} = \frac{2qL^2}{5}$$
$$V_B = \frac{16qL}{25} - \frac{qL}{5} = \frac{11qL}{25}$$
$$M_B = -\frac{qL^2}{3} + \frac{qL^2}{3} = 0$$

Solución 2

Estructura equivalente

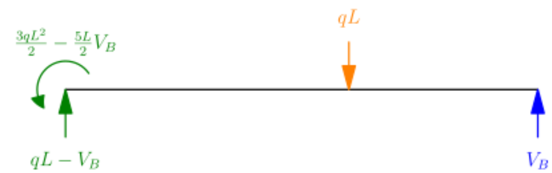


Suma de fuerzas y momentos

$$V_A - qL + V_B = 0$$
$$M_A - qL\left(\frac{3L}{2}\right) + V_B\left(\frac{5L}{2}\right) = 0$$
$$M_A - V_A\left(\frac{5L}{2}\right) + qL(L) = 0$$

No tomo en cuenta la última ecuación, despejo V_A y M_A

$$V_A = qL - V_B$$
$$M_A = \frac{3qL^2}{2} - \frac{5L}{2}V_B$$



Momento de $0 \leqslant x \leqslant L$

$$M = -M_A + V_A x = -\frac{3qL^2}{2} + \frac{5L}{2}V_B + (qL - V_B)x$$

Momento de $L \leqslant x \leqslant 2L$

$$M = -M_A + V_A x - \frac{q}{2}(x-L)^2 = -2qL^2 + \frac{5L}{2}V_B + 2qLx - V_B x - \frac{q}{2}x^2$$

Momento de $\frac{L}{2} \geqslant x \geqslant 0$

$$M = V_B x$$

Energía de deformación por flexión

$$U_i = \int_0^L \frac{M^2}{2EI}\,dx + \int_L^{2L} \frac{M^2}{2EI}\,dx + \int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dx$$

Reemplazando

$$\begin{aligned} U_i = &\frac{1}{2EI}\int_0^L \left[-\frac{3qL^2}{2} + \frac{5L}{2}V_B + (qL - V_B)x\right]^2 dx \\ &+ \frac{1}{2EI}\int_L^{2L} \left(-2qL^2 + \frac{5L}{2}V_B + 2qLx - V_B x - \frac{q}{2}x^2\right)^2 dx \\ &+ \frac{1}{2EI}\int_0^{\frac{L}{2}} (V_B x)^2 dx \end{aligned}$$

Integrando

$$U_i = \frac{L^3}{240EI}\left(136q^2L^2 - 550qLV_B + 625V_B^2\right)$$

Minimizando

$$\frac{dU_i}{dV_B} = -\frac{5L^3}{24EI}\Big(11qL - 25V_B\Big) = 0$$

Despejando V_B

$$V_B = \frac{11qL}{25}$$

Reemplazando en las demás reacciones

$$V_A = qL - V_B = qL - \frac{11qL}{25} = \frac{14qL}{25}$$

$$M_A = \frac{3qL^2}{2} - \frac{5L}{2}V_B = \frac{3qL^2}{2} - \frac{5L}{2}\left(\frac{11qL}{25}\right) = \frac{2qL^2}{5}$$