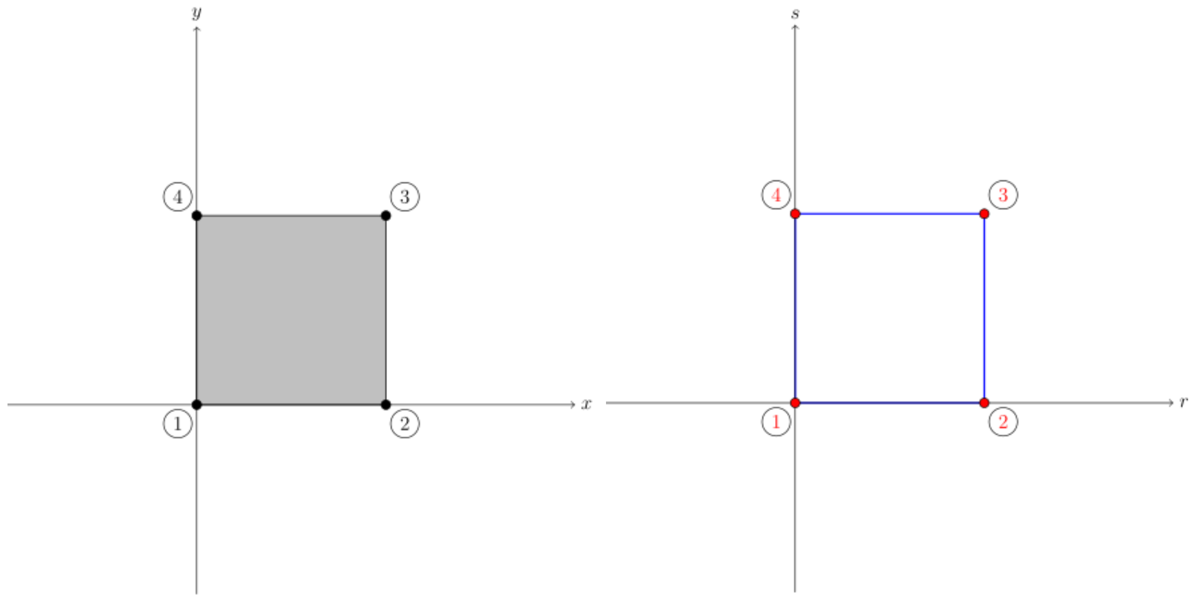
# Introducción a elementos finitos
# Tarea 9 I-2017

Calcular la matriz de rigidez de la placa de acero A36 con dimensiones 250 mm × 250 mm, espesor 20 mm, coeficiente de Poisson 0.26 y módulo de elasticidad 200 GPa sujeta a esfuerzo plano, usando un elemento en coordenadas naturales




### Solución 1

La matriz de rigidez es

$$K = \int_0^1 \int_0^1 B^{\mathrm{T}}\, C\, B\, \det J\, t\, ds\, dr$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{200000}{1-0.26^2} \begin{bmatrix} 1 & 0.26 & 0 \\ 0.26 & 1 & 0 \\ 0 & 0 & \frac{1-0.26}{2} \end{bmatrix} = \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix}$$

Coordenadas de los nodos del elemento lagrangiano

$$\textcircled{1} = [r_1, s_1] = [0, 0] \quad \textcircled{3} = [r_3, s_3] = [1, 1]$$
$$\textcircled{2} = [r_2, s_2] = [1, 0] \quad \textcircled{4} = [r_4, s_4] = [0, 1]$$

Funciones que interpolan los desplazamientos

$$\begin{aligned}
N_1 &= \frac{r - r_2}{r_1 - r_2} \cdot \frac{s - s_4}{s_1 - s_4} = \frac{r - 1}{0 - 1} \cdot \frac{s - 1}{0 - 1} = (r - 1)(s - 1) \\
N_2 &= \frac{r - r_1}{r_2 - r_1} \cdot \frac{s - s_3}{s_2 - s_3} = \frac{r - 0}{1 - 0} \cdot \frac{s - 1}{0 - 1} = -r(s - 1) \\
N_3 &= \frac{r - r_4}{r_3 - r_4} \cdot \frac{s - s_2}{s_3 - s_2} = \frac{r - 0}{1 - 0} \cdot \frac{s - 0}{1 - 0} = rs \\
N_4 &= \frac{r - r_3}{r_4 - r_3} \cdot \frac{s - s_1}{s_4 - s_1} = \frac{r - 1}{0 - 1} \cdot \frac{s - 0}{1 - 0} = -s(r - 1)
\end{aligned}$$

Escribiendo en la forma matricial

$$N = \begin{bmatrix} (r-1)(s-1) & -r(s-1) & rs & -s(r-1) \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

Coordenadas de los nodos de la placa

$$\textcircled{1} = [x_1, y_1] = [0, 0] \quad \textcircled{3} = [x_3, y_3] = [250, 250]$$
$$\textcircled{2} = [x_2, y_2] = [250, 0] \quad \textcircled{4} = [x_4, y_4] = [0, 250]$$

Funciones que interpolan la geometría

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Reemplazando las coordenadas de los nodos

$$x = 250r$$
$$y = 250s$$

El jacobiano y el jacobiano inverso son

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix}$$

Reemplazando derivadas

$$J = \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{1}{250} & 0 \\ 0 & \frac{1}{250} \end{bmatrix}$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

Debido a que las funciones de forma están en función de $r$ y $s$, se usará la regla de la cadena

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial x}$$
$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial y}$$

Reemplazando en $B_i$

$$B_1 = \begin{bmatrix} \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{s-1}{250} & 0 \\ 0 & \frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{s-1}{250} & 0 \\ 0 & -\frac{r}{250} \\ -\frac{r}{250} & -\frac{s-1}{250} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{s}{250} & 0 \\ 0 & \frac{r}{250} \\ \frac{r}{250} & \frac{s}{250} \end{bmatrix}$$

$$B_4 = \begin{bmatrix} \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{s}{250} & 0 \\ 0 & -\frac{r-1}{250} \\ -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix}$$

Reemplazando en $B$

$$B = \begin{bmatrix} \frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0 \\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K = \int_0^1 \int_0^1 \begin{bmatrix} \frac{s-1}{250} & 0 & \frac{r-1}{250} \\ 0 & \frac{r-1}{250} & \frac{s-1}{250} \\ -\frac{s-1}{250} & 0 & -\frac{r}{250} \\ 0 & -\frac{r}{250} & -\frac{s-1}{250} \\ \frac{s}{250} & 0 & \frac{r}{250} \\ 0 & \frac{r}{250} & \frac{s}{250} \\ -\frac{s}{250} & 0 & -\frac{r-1}{250} \\ 0 & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} \frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0 \\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix} 62500 \cdot 20 \, ds \, dr$$

Integrando

$$K = \begin{bmatrix} 1959101.959 & 675675.6755 & -1165451.165 & -117975.1180 & -979550.9793 & -675675.6757 & 185900.1856 & 117975.1178 \\ 675675.6755 & 1959101.960 & 117975.1180 & 185900.1852 & -675675.6756 & -979550.9795 & -117975.1181 & -1165451.166 \\ -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755 & 185900.1856 & -117975.1178 & -979550.9793 & 675675.6757 \\ -117975.1180 & 185900.1852 & -675675.6755 & 1959101.960 & 117975.1181 & -1165451.166 & 675675.6756 & -979550.9795 \\ -979550.9796 & -675675.6756 & 185900.1859 & 117975.1181 & 1959101.959 & 675675.6755 & -1165451.165 & -117975.1180 \\ -675675.6757 & -979550.9797 & -117975.1178 & -1165451.165 & 675675.6755 & 1959101.959 & 117975.1180 & 185900.1860 \\ 185900.1859 & -117975.1181 & -979550.9796 & 675675.6756 & -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755 \\ 117975.1178 & -1165451.165 & 675675.6757 & -979550.9797 & -117975.1180 & 185900.1860 & -675675.6755 & 1959101.959 \end{bmatrix}$$

**Solución 2**

La matriz de rigidez es

$$K = \int_0^1 \int_0^1 B^{\mathrm{T}} \, C \, B \, \det J \, t \, ds \, dr$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{200000}{1-0.26^2} \begin{bmatrix} 1 & 0.26 & 0 \\ 0.26 & 1 & 0 \\ 0 & 0 & \frac{1-0.26}{2} \end{bmatrix} = \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix}$$

Coordenadas de los nodos del elemento lagrangiano

$$\textcircled{1} = [r_1, s_1] = [0, 0] \quad \textcircled{3} = [r_3, s_3] = [1, 1]$$
$$\textcircled{2} = [r_2, s_2] = [1, 0] \quad \textcircled{4} = [r_4, s_4] = [0, 1]$$

Funciones que interpolan los desplazamientos

$$
\begin{aligned}
N_1 &= \frac{r-r_2}{r_1-r_2}\cdot\frac{s-s_4}{s_1-s_4} = \frac{r-1}{0-1}\cdot\frac{s-1}{0-1} = (r-1)(s-1)\\
N_2 &= \frac{r-r_1}{r_2-r_1}\cdot\frac{s-s_3}{s_2-s_3} = \frac{r-0}{1-0}\cdot\frac{s-1}{0-1} = -r(s-1)\\
N_3 &= \frac{r-r_4}{r_3-r_4}\cdot\frac{s-s_2}{s_3-s_2} = \frac{r-0}{1-0}\cdot\frac{s-0}{1-0} = rs\\
N_4 &= \frac{r-r_3}{r_4-r_3}\cdot\frac{s-s_1}{s_4-s_1} = \frac{r-1}{0-1}\cdot\frac{s-0}{1-0} = -s(r-1)
\end{aligned}
$$

Escribiendo en la forma matricial

$$
N = \begin{bmatrix}(r-1)(s-1) & -r(s-1) & rs & -s(r-1)\end{bmatrix} = \begin{bmatrix}N_1 & N_2 & N_3 & N_4\end{bmatrix}
$$

Coordenadas de los nodos de la placa

$$
\begin{aligned}
&① = [x_1, y_1] = [0, 0] && ③ = [x_3, y_3] = [250, 250]\\
&② = [x_2, y_2] = [250, 0] && ④ = [x_4, y_4] = [0, 250]
\end{aligned}
$$

Funciones que interpolan la geometría

$$
\begin{aligned}
x &= N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4\\
y &= N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4
\end{aligned}
$$

Reemplazando las coordenadas de los nodos

$$
\begin{aligned}
x &= 250r\\
y &= 250s
\end{aligned}
$$

El jacobiano y el jacobiano inverso son

$$
J = \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r}\\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s}\end{bmatrix} \quad J^{-1} = \begin{bmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y}\end{bmatrix}
$$

Reemplazando derivadas

$$
J = \begin{bmatrix}250 & 0\\ 0 & 250\end{bmatrix} \quad J^{-1} = \begin{bmatrix}\frac{1}{250} & 0\\ 0 & \frac{1}{250}\end{bmatrix}
$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

Las derivadas de las funciones de forma se calcularán en forma matricial

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix}$$

Reemplazando en $N_i$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial r} \\ \frac{\partial N_1}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{1}{250} & 0 \\ 0 & \frac{1}{250} \end{bmatrix} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{250} \\ \frac{r-1}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial r} \\ \frac{\partial N_2}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{1}{250} & 0 \\ 0 & \frac{1}{250} \end{bmatrix} \begin{bmatrix} -(s-1) \\ -r \end{bmatrix} = \begin{bmatrix} -\frac{s-1}{250} \\ -\frac{r}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{1}{250} & 0 \\ 0 & \frac{1}{250} \end{bmatrix} \begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} \frac{s}{250} \\ \frac{r}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_4}{\partial x} \\ \frac{\partial N_4}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_4}{\partial r} \\ \frac{\partial N_4}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{1}{250} & 0 \\ 0 & \frac{1}{250} \end{bmatrix} \begin{bmatrix} -s \\ -(r-1) \end{bmatrix} = \begin{bmatrix} -\frac{s}{250} \\ -\frac{r-1}{250} \end{bmatrix}$$

Reemplazando en $B$

$$B = \begin{bmatrix} \frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0 \\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K=\int_0^1\int_0^1\begin{bmatrix}\frac{s-1}{250} & 0 & \frac{r-1}{250}\\ 0 & \frac{r-1}{250} & \frac{s-1}{250}\\ -\frac{s-1}{250} & 0 & -\frac{r}{250}\\ 0 & -\frac{r}{250} & -\frac{s-1}{250}\\ \frac{s}{250} & 0 & \frac{r}{250}\\ 0 & \frac{r}{250} & \frac{s}{250}\\ -\frac{s}{250} & 0 & -\frac{r-1}{250}\\ 0 & -\frac{r-1}{250} & -\frac{s}{250}\end{bmatrix}\begin{bmatrix}214500.214 & 55770.056 & 0\\ 55770.056 & 214500.214 & 0\\ 0 & 0 & 79365.079\end{bmatrix}\begin{bmatrix}\frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0\\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250}\\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250}\end{bmatrix}62500\cdot 20\,ds\,dr$$

Integrando

$$K=\begin{bmatrix}1959101.959 & 675675.6755 & -1165451.165 & -117975.1180 & -979550.9793 & -675675.6757 & 185900.1856 & 117975.1178\\ 675675.6755 & 1959101.960 & 117975.1180 & 185900.1852 & -675675.6756 & -979550.9795 & -117975.1181 & -1165451.166\\ -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755 & 185900.1856 & -117975.1178 & -979550.9793 & 675675.6757\\ -117975.1180 & 185900.1852 & -675675.6755 & 1959101.960 & 117975.1181 & -1165451.166 & 675675.6756 & -979550.9795\\ -979550.9796 & -675675.6756 & 185900.1859 & 117975.1181 & 1959101.959 & 675675.6755 & -1165451.165 & -117975.1180\\ -675675.6757 & -979550.9797 & -117975.1178 & -1165451.165 & 675675.6755 & 1959101.959 & 117975.1180 & 185900.1860\\ 185900.1859 & -117975.1181 & -979550.9796 & 675675.6756 & -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755\\ 117975.1178 & -1165451.165 & 675675.6757 & -979550.9797 & -117975.1180 & 185900.1860 & -675675.6755 & 1959101.959\end{bmatrix}$$

**Solución 3**

La matriz de rigidez es

$$K=\int_0^1\int_0^1 B^{\mathrm{T}}\,C\,B\,\det J\,t\,ds\,dr$$

La matriz constitutiva es

$$C=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}=\frac{200000}{1-0.26^2}\begin{bmatrix}1 & 0.26 & 0\\ 0.26 & 1 & 0\\ 0 & 0 & \frac{1-0.26}{2}\end{bmatrix}=\begin{bmatrix}214500.214 & 55770.056 & 0\\ 55770.056 & 214500.214 & 0\\ 0 & 0 & 79365.079\end{bmatrix}$$

Coordenadas de los nodos del elemento lagrangiano

$$\text{①}=[r_1,s_1]=[0,0]\quad \text{③}=[r_3,s_3]=[1,1]$$
$$\text{②}=[r_2,s_2]=[1,0]\quad \text{④}=[r_4,s_4]=[0,1]$$

Funciones que interpolan los desplazamientos

$$
\begin{aligned}
N_1 &= \frac{r-r_2}{r_1-r_2}\cdot\frac{s-s_4}{s_1-s_4} = \frac{r-1}{0-1}\cdot\frac{s-1}{0-1} = (r-1)(s-1) \\
N_2 &= \frac{r-r_1}{r_2-r_1}\cdot\frac{s-s_3}{s_2-s_3} = \frac{r-0}{1-0}\cdot\frac{s-1}{0-1} = -r(s-1) \\
N_3 &= \frac{r-r_4}{r_3-r_4}\cdot\frac{s-s_2}{s_3-s_2} = \frac{r-0}{1-0}\cdot\frac{s-0}{1-0} = rs \\
N_4 &= \frac{r-r_3}{r_4-r_3}\cdot\frac{s-s_1}{s_4-s_1} = \frac{r-1}{0-1}\cdot\frac{s-0}{1-0} = -s(r-1)
\end{aligned}
$$

Escribiendo en la forma matricial

$$
N = \begin{bmatrix}(r-1)(s-1) & -r(s-1) & rs & -s(r-1)\end{bmatrix} = \begin{bmatrix}N_1 & N_2 & N_3 & N_4\end{bmatrix}
$$

Coordenadas de los nodos de la placa

$$
\begin{aligned}
&① = [x_1, y_1] = [0,0] && ③ = [x_3, y_3] = [250,250] \\
&② = [x_2, y_2] = [250,0] && ④ = [x_4, y_4] = [0,250]
\end{aligned}
$$

Funciones que interpolan la geometría

$$
\begin{aligned}
x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \\
y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4
\end{aligned}
$$

Reemplazando las coordenadas de los nodos

$$
\begin{aligned}
x &= 250r \\
y &= 250s
\end{aligned}
$$

El jacobiano y el jacobiano inverso son

$$
J = \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s}\end{bmatrix} \quad J^{-1} = \begin{bmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y}\end{bmatrix}
$$

Reemplazando derivadas

$$
J = \begin{bmatrix}250 & 0 \\ 0 & 250\end{bmatrix} \quad J^{-1} = \begin{bmatrix}\frac{1}{250} & 0 \\ 0 & \frac{1}{250}\end{bmatrix}
$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

Las derivadas de las funciones de forma se calcularán usando una forma alternativa del jacobiano inverso

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix}$$

Reemplazando en $N_i$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial r} \\ \frac{\partial N_1}{\partial s} \end{bmatrix} = \frac{1}{62500} \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{250} \\ \frac{r-1}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial r} \\ \frac{\partial N_2}{\partial s} \end{bmatrix} = \frac{1}{62500} \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix} \begin{bmatrix} -(s-1) \\ -r \end{bmatrix} = \begin{bmatrix} -\frac{s-1}{250} \\ -\frac{r}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial s} \end{bmatrix} = \frac{1}{62500} \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix} \begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} \frac{s}{250} \\ \frac{r}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_4}{\partial x} \\ \frac{\partial N_4}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_4}{\partial r} \\ \frac{\partial N_4}{\partial s} \end{bmatrix} = \frac{1}{62500} \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix} \begin{bmatrix} -s \\ -(r-1) \end{bmatrix} = \begin{bmatrix} -\frac{s}{250} \\ -\frac{r-1}{250} \end{bmatrix}$$

Reemplazando en $B$

$$B = \begin{bmatrix} \frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0 \\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K = \int_0^1 \int_0^1 \begin{bmatrix} \frac{s-1}{250} & 0 & \frac{r-1}{250} \\ 0 & \frac{r-1}{250} & \frac{s-1}{250} \\ -\frac{s-1}{250} & 0 & -\frac{r}{250} \\ 0 & -\frac{r}{250} & -\frac{s-1}{250} \\ \frac{s}{250} & 0 & \frac{r}{250} \\ 0 & \frac{r}{250} & \frac{s}{250} \\ -\frac{s}{250} & 0 & -\frac{r-1}{250} \\ 0 & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} \frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0 \\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix} 62500 \cdot 20 \, ds \, dr$$

Integrando

$$K = \begin{bmatrix} 1959101.959 & 675675.6755 & -1165451.165 & -117975.1180 & -979550.9793 & -675675.6757 & 185900.1856 & 117975.1178 \\ 675675.6755 & 1959101.960 & 117975.1180 & 185900.1852 & -675675.6756 & -979550.9795 & -117975.1181 & -1165451.166 \\ -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755 & 185900.1856 & -117975.1178 & -979550.9793 & 675675.6757 \\ -117975.1180 & 185900.1852 & -675675.6755 & 1959101.960 & 117975.1181 & -1165451.166 & 675675.6756 & -979550.9795 \\ -979550.9796 & -675675.6756 & 185900.1859 & 117975.1181 & 1959101.959 & 675675.6755 & -1165451.165 & -117975.1180 \\ -675675.6757 & -979550.9797 & -117975.1178 & -1165451.165 & 675675.6755 & 1959101.959 & 117975.1180 & 185900.1860 \\ 185900.1859 & -117975.1181 & -979550.9796 & 675675.6756 & -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755 \\ 117975.1178 & -1165451.165 & 675675.6757 & -979550.9797 & -117975.1180 & 185900.1860 & -675675.6755 & 1959101.959 \end{bmatrix}$$

**Solución 4**

La matriz de rigidez es

$$K = \int_0^1 \int_0^1 B^{\mathrm{T}} \, C \, B \, \det J \, t \, ds \, dr$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{200000}{1-0.26^2} \begin{bmatrix} 1 & 0.26 & 0 \\ 0.26 & 1 & 0 \\ 0 & 0 & \frac{1-0.26}{2} \end{bmatrix} = \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix}$$

Coordenadas de los nodos del elemento lagrangiano

$$\textcircled{1} = [r_1, s_1] = [0, 0] \quad \textcircled{3} = [r_3, s_3] = [1, 1]$$
$$\textcircled{2} = [r_2, s_2] = [1, 0] \quad \textcircled{4} = [r_4, s_4] = [0, 1]$$

Funciones que interpolan los desplazamientos

$$N_1 = \frac{r-r_2}{r_1-r_2} \cdot \frac{s-s_4}{s_1-s_4} = \frac{r-1}{0-1} \cdot \frac{s-1}{0-1} = (r-1)(s-1)$$
$$N_2 = \frac{r-r_1}{r_2-r_1} \cdot \frac{s-s_3}{s_2-s_3} = \frac{r-0}{1-0} \cdot \frac{s-1}{0-1} = -r(s-1)$$
$$N_3 = \frac{r-r_4}{r_3-r_4} \cdot \frac{s-s_2}{s_3-s_2} = \frac{r-0}{1-0} \cdot \frac{s-0}{1-0} = rs$$
$$N_4 = \frac{r-r_3}{r_4-r_3} \cdot \frac{s-s_1}{s_4-s_1} = \frac{r-1}{0-1} \cdot \frac{s-0}{1-0} = -s(r-1)$$

Escribiendo en la forma matricial

$$N = \begin{bmatrix} (r-1)(s-1) & -r(s-1) & rs & -s(r-1) \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

Coordenadas de los nodos de la placa

$$\textcircled{1} = [x_1, y_1] = [0, 0] \qquad \textcircled{3} = [x_3, y_3] = [250, 250]$$
$$\textcircled{2} = [x_2, y_2] = [250, 0] \qquad \textcircled{4} = [x_4, y_4] = [0, 250]$$

Funciones que interpolan la geometría

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Reemplazando las coordenadas de los nodos

$$x = 250r$$
$$y = 250s$$

El jacobiano y el jacobiano inverso son

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix}$$

Reemplazando derivadas

$$J = \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{1}{250} & 0 \\ 0 & \frac{1}{250} \end{bmatrix}$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = M_1\, M_2\, M_3$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} & 0 & 0 \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ 0 & 0 & \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 & \frac{\partial N_4}{\partial r} & 0 \\ \frac{\partial N_1}{\partial s} & 0 & \frac{\partial N_2}{\partial s} & 0 & \frac{\partial N_3}{\partial s} & 0 & \frac{\partial N_4}{\partial s} & 0 \\ 0 & \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 & \frac{\partial N_4}{\partial r} \\ 0 & \frac{\partial N_1}{\partial s} & 0 & \frac{\partial N_2}{\partial s} & 0 & \frac{\partial N_3}{\partial s} & 0 & \frac{\partial N_4}{\partial s} \end{bmatrix}$$

Reemplazando en $M_2$ y $M_3$

$$M_2 = \begin{bmatrix} \frac{1}{250} & 0 & 0 & 0 \\ 0 & \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} & 0 \\ 0 & 0 & 0 & \frac{1}{250} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} s-1 & 0 & -(s-1) & 0 & s & 0 & -s & 0 \\ r-1 & 0 & -r & 0 & r & 0 & -(r-1) & 0 \\ 0 & s-1 & 0 & -(s-1) & 0 & s & 0 & -s \\ 0 & r-1 & 0 & -r & 0 & r & 0 & -(r-1) \end{bmatrix}$$

Reemplazando en $B$

$$B = \begin{bmatrix} \frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0 \\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K = \int_0^1 \int_0^1 \begin{bmatrix} \frac{s-1}{250} & 0 & \frac{r-1}{250} \\ 0 & \frac{r-1}{250} & \frac{s-1}{250} \\ -\frac{s-1}{250} & 0 & -\frac{r}{250} \\ 0 & -\frac{r}{250} & -\frac{s-1}{250} \\ \frac{s}{250} & 0 & \frac{r}{250} \\ 0 & \frac{r}{250} & \frac{s}{250} \\ -\frac{s}{250} & 0 & -\frac{r-1}{250} \\ 0 & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} \frac{s-1}{250} & 0 & -\frac{s-1}{250} & 0 & \frac{s}{250} & 0 & -\frac{s}{250} & 0 \\ 0 & \frac{r-1}{250} & 0 & -\frac{r}{250} & 0 & \frac{r}{250} & 0 & -\frac{r-1}{250} \\ \frac{r-1}{250} & \frac{s-1}{250} & -\frac{r}{250} & -\frac{s-1}{250} & \frac{r}{250} & \frac{s}{250} & -\frac{r-1}{250} & -\frac{s}{250} \end{bmatrix} 62500 \cdot 20 \, ds \, dr$$

Integrando

$$K = \begin{bmatrix} 1959101.959 & 675675.6755 & -1165451.165 & -117975.1180 & -979550.9793 & -675675.6757 & 185900.1856 & 117975.1178 \\ 675675.6755 & 1959101.960 & 117975.1180 & 185900.1852 & -675675.6756 & -979550.9795 & -117975.1181 & -1165451.166 \\ -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755 & 185900.1856 & -117975.1178 & -979550.9793 & 675675.6757 \\ -117975.1180 & 185900.1852 & -675675.6755 & 1959101.960 & 117975.1181 & -1165451.166 & 675675.6756 & -979550.9795 \\ -979550.9796 & -675675.6756 & 185900.1859 & 117975.1181 & 1959101.959 & 675675.6755 & -1165451.165 & -117975.1180 \\ -675675.6757 & -979550.9797 & -117975.1178 & -1165451.165 & 675675.6755 & 1959101.959 & 117975.1180 & 185900.1860 \\ 185900.1859 & -117975.1181 & -979550.9796 & 675675.6756 & -1165451.165 & 117975.1180 & 1959101.959 & -675675.6755 \\ 117975.1178 & -1165451.165 & 675675.6757 & -979550.9797 & -117975.1180 & 185900.1860 & -675675.6755 & 1959101.959 \end{bmatrix}$$